week4lecture1

Today we will continue with the See it chapter and then on Thursday we will start the Simplify it chapter. After this we will learn a couple of useful counting techniques commonly used in mathematics. So for today we will go over the following problem together and then do a variation. For the variation, I will have you partner up with one other person, write up a clear solution on your paper, and two people will write their solutions on the board, verbatim. This way we can again work on the communication aspect of problem solving.

Acrobats You are a manager of a troupe of acrobats. Each of your acrobats wears an outfit consisting of a top and tights. Both the tops and tights come in blue, red, and yellow and each acrobat wears a different color scheme. If all possible combinations appear (and non are repeated), how many acrobats are in the troupe?

Next, the king wants to hire a royal team of acrobats from your troupe-three acrobats wearing tops that are all the same or of each color, and tights that are all the same or of each color. He agrees to let you send one acrobat at a time to the throne room, where they remain until he can select a royal team from the acrobats gathered there. If he pays you \$100 for each acrobat you send over, how much money can you make?

Acrobats variation Suppose the tops and tights now come in blue, red, yellow, and green and each acrobat wears a different color scheme. Next, the king wants to hire a royal team of acrobats from your troupe-four acrobats wearing tops that are all the same or of each color, and tights that are all the same or of each color. He has the same agreement. How much money can you make?

So we see that we can make at most \$500 and at least \$300. Next we will go over some counting techniques and then do the variation of the problem.

Permutations

Today we will cover a useful mathematical technique used in counting. We will start off by defining what a *Permutation* is and then seeing how we may use it. So, what is a permutation?

Definition 1. ***Permutation*** *Given an ordered list of things, a permutation on that list is a rearrangement of the things.*

Examples 1. *Say our list of things is* $(1, 2, 3)$*. Let's list out all the possible permutations. One permutation is* $(2, 1, 3)$*. How about another? Another might be* $(3, 2, 1)$*. Some more are* $(1, 3, 2)$*,* $(3, 1, 2)$*,* $(2, 3, 1)$*, and* $(1, 2, 3)$*. Note that we do in fact consider doing nothing as a permutation. We call this the identity permutation.*

We may generalize this example and ask a question. **How many ways can we rearrange a list of** n **things?** That is, how many ways can we permute a list? To answer this question, we will see if we can determine a pattern using a table, much like we did in the Green's party problem. But before we go on, let me remind you the definition of a factorial.

Definition 2. ***Factorial*** *We define the factorial of a positive whole number* n *by* $n! = n(n-1)(n-2)\cdots 321 = n!$.

Examples 2. $5! = (5)(4)(3)(2)(1) = 120$

$6! = (6)(5!) = 720$

Try to keep this in mind for what we do next. So let's form a table and compute the number of permutations for lists of different sizes.

1	1
2	2
3	6
4	24
n	n!

I conjecture that the number of permutations on a list of n things is $n!$. Let's try to understand this. Suppose we have 3 things. We can move the first thing in three ways: swap it with itself, swap it with the second thing, swap it with the third thing. Label these as (11)=swap with itself, (12) swap 1 with 2, (13) swap 1 with 3. We can move the second thing in three ways as well, swap it with itself, swap it with the first thing, swap it with the third thing. Label these as (22)=swap with itself, (21) swap 2 with 1, (23) swap 2 with 3. Notice (21) and (12) do the same thing and hence they don't count as doing different things. So (12) = (21). Similarly, we only get one uncounted swap for the last item; namely, (33). **To get a permutation, we combine these swappings**. By this, I mean, we have the following possible permutations:

(11)	(22)	(33)
(11)	(23)	(33)
(12)	(22)	(33)
(12)	(23)	(33)
(13)	(22)	(33)
(13)	(23)	(33)

Let's look at what each of these permutations does on the list (123).
Show what each permutation does
So what is going on here? Let's look at this informally. Suppose we have a list of n things. We may move the first thing n ways. We may move the second thing $n-1$ ways, this is because swapping the first and second thing is already counted. It follows that the mth thing can be moved $n-m+1$ ways. Therefore, there are $n(n-1)(n-2)\cdots(3)(2)(1) = n!$ ways to permute n things.
Let's give two examples. Given the word $HAMBURGER$, how many ways can we rearrange the letters to form more nonsensical words? How about the word $HOOT$? We see there is an issue here because we have two O's. To account for this, we divide by the number of ways we can swap the $O's$ since swapping them won't change anything. Since there are 2 $O's$, we simply divide by 2.
Combinations The next technique we will consider is commonly used in finding the number of combinations possible in certain problems.

Examples 3. *There are 4 ice cream flavors. An ice cream cone can have at most one scoop of each flavor and must have exactly 2 scoops. How many possible flavor combinations are there?*

Solution We see that we are effectively trying to find the number of ways we can choose 2 flavors out of 4. To solve this problem and problems like it we use a function called the choose function. In general, if we have n objects and we choose k of them, the number of ways we can do this is written

$$n\,choose\,k = \frac{n!}{(n-k)!k!}$$

So for our case, it's written $4\,choose\,2 = \frac{4!}{(4-2)!2!} = 2\cdot 3 = 6$. Therefore, there are 6 different combinations. So why is this important? We will see problems that use this counting technique, but I have a funny anecdote. A friend of one of my old professors was approached by an ice cream company (something like Baskin Robins), and was asked to compute the number of ice cream combinations with some extra conditions. They offered him $80 an hour to do so (consulting as a mathematician can get you that sort of money). Lucky for him, the company had no idea the problem was trivial and was easily computed using something like the choose function. So he sat around for 8 hours, did a simple computation, and made bank. A good lesson to take from this is that if you are ignorant of mathematics, you can get ripped off for a lot of money.
Counting first n integers Next we consider something I went over before. I will show you again how to count the first n positive numbers. We will need to know how to do this in a problem you will see soon. *Show Gauss trick*
Now we will go over a couple of counting techniques used in mathematics and that will be used in this course. We will go over counting permutations, counting combinations, and adding the first n integers, which I briefly showed you before. We will derive formulas for these techniques, but if you ever need them on the exam or homework, I will provide them. The important part will be understanding what

they mean and how to use them and a common way to acquire this understanding is both by using them and by deriving them. So we will start off with permutations.