Oct13 Lecture Notes

1 Stretches

So we will start doing stretches again. We will do this for the following reasons:

1. For fun!
2. Get our thinking muscles stretched.
3. Some problems are genuinely cool.
4. Practice thinking outside of the box.

So the stretch structure will be:

- Do problems on your own for 5 minutes.
- Go over them as a class.
- Forget them forever.

So let's start off doing the following two stretches:

1. Imagine you are a mathematician given the task to design the very first manhole cover. What shape would you use and why? Try to use as much mathematical reasoning as possible. Questions of this type arise all the time in the real world.
2. Add one line to IX to get an expression equal to 6.

Here are the solutions:

1. My first thought was the following. Given a certain desired "width" for the manhole, a triangle is going to be too cramped, and a circle will require less material than a square. To see this, construct a square with width a. A circle with width, or rather, diameter a, can fit entirely inside the square.
2. The solution is quite silly, but simply write an "S" in front of the "IX" to get "SIX". Notice I never said the line must absolutely be a straight line.

2 Finding the price

This week we will continue the guess and check theme we have been studying.

Example 1. When a farm burned, they found a bill: 72 turkeys, \$_67.9_. The fire smudge obliterated the first and last digit of the total price of the birds. What are the two faded digits and what was the price of a single turkey?

Solution 1.

1. We need to grasp the problem and disambiguate things.
   - We assume a turkey cannot cost a fraction of a penny: e.g., a turkey cannot cost \$10.025.
   - We assume each turkey costs the same amount.
2. What sort of value can go in the last digit place? Note by the 2 in 72, we must have the last digit in the price be a multiple of 2.
   - To see this, try multiplying out 72 by decimal numbers. We will see that the last digit will always be a multiple of 2.
3. To simplify things, let's guess the last digit is 0. Note: other even numbers will work.

4. Here we guess and check by the following process.
   (a) Guess the first digit is 1 to get \$167.90 for the total price.
   (b) Divide the price by 72 to get an approximate price per turkey: \$2.33194444....
   (c) Each turkey must cost either \$2.33 or \$2.34 since we cannot use fractional pennies.
   (d) We multiply this price per turkey by 72 and check whether or not this price per turkey works.
   (e) Assuming the price is \$2.33 per turkey, we get $72 \times \$2.33 = \$167.76$. Notice the dime's place is not a 9, so this does not work.
   (f) Assuming the price is \$2.34 per turkey, we get $72 \times \$2.34 = \$168.48$ for similar reasons. Again, this does not work.
   (g) Before we go on, we note that if we tried \$167.92 or \$167,94, or any other even number, when dividing by 72, we always get approximately \$2.33 per turkey. This is why using zero doesn't make a difference.
5. We see that guessing the first digit to be 1 does not work. We will do a couple of more guesses to convey the idea.
6. Let's guess the first digit is 2 to get \$267.90.
   (a) Divide by 72. Approximate turkey price: \$3.72.
   (b) Each turkey must either be \$3.72 or \$3.73.
   (c) We thus find total cost is $72 \times \$3.72 = \$267.84$ or $72 \times \$3.73 = \$268.56$, neither of which work.
7. Let's do one more guess. Let's guess the first digit to be 3 to get \$367.90.
   (a) Divide by 72. Approximate turkey price: \$5.1097222222....
   (b) Each turkey must either be \$5.10 or \$5.11.
   (c) We thus find total cost is $72 \times \$5.10 = \$367$ or $72 \times \$367.92$.
8. What do you know. We guessed correctly!

We shall go over two more similar examples to really solidify the method here.

Example 2. When a fence burned, they found a bill: 65 goats, \$_614.9_. Find the price of a single goat.

Solution 2. We make similar assumptions as before: A goat cannot cost a fraction of a penny and each goat costs the same.

1. We note that since we have 65 goats, the last digit must either be a 0 or a 5.
2. We guess the first digit like before.
   (a) Guess the first digit is 1 to get \$1614.90.
   (b) Divide by 65. Approximate goat price: \$24.84461....
   (c) Each goat must either be \$24.84 or \$24.85.
   (d) Thus total cost is $65 \times \$24.84 = \$1614.6$ or $65 \times \$24.85 = \$1615.15$.
   (e) Neither work.
3. We do another guess.
   (a) Guess the first digit is 2 to get \$2614.90.
   (b) Divide by 65. Approximate goat price: \$40.229230....
   (c) Each goat must either be \$40.22 or \$40.23.
   (d) Thus total cost is $65 \times \$40.22 = \$2614.3$ or $65 \times \$40.23 = \$2614.95$.

4. Awesome, got it!

Okay, let's do one more example.

Example 3. Turns out you were the one who bought the 65 goats. Unfortunately, one goat was rabid and bit apart a bill that now read: 52 kazoos, \$_20.8_. Find the price of a single kazoo.

Solution 3. Make the similar assumptions as before.

1. We know the last digit must be a 0 because there were 52 kazoos.
2. We guess the first digit like before.
   (a) Guess the first digit to be 1 to get \$120.80.
   (b) Divide by 52. Approximate kazoo price: \$2.3238....
   (c) Each kazoo must either be \$2.32 or \$2.33.
   (d) Thus total cost is $52 \times \$2.32 = \$120.64$ or $52 \times \$2.33 = 121.16$.
   (e) Neither work.
3. We guess again.
   (a) Guess first digit is 2 to get \$220.80.
   (b) Divide by 52. Approximate kazoo price: \$4.2461....
   (c) Each kazoo must either be \$4.24 or \$4.25.
   (d) Total cost is $52 \times \$4.24 = \$220.48$ or $52 \times \$4.25 = \$221$.
   (e) Neither work.
4. Let's guess once more!
   (a) Guess first digit is 3 to get \$320.80.
   (b) Divide by 52. Approximate kazoo price: \$6.16 or \$6.17.
   (c) Total cost is $52 \times \$6.16 = \$320.32$ or $52 \times \$6.17 = \$320.84$.
   (d) Bingo!