1 Introduction

I wanted to remind you that a lot of information going to be presented is mostly going to serve a single purpose. The purpose is the following. In order to talk about mathematical ideas that are actually interesting, some background knowledge and intuition is needed. This is an especially difficult task to attempt for a class of students with little to no background in pure mathematics. As a consequence, you will not be responsible for a lot of the material, but rather the ability to use the material. So, for example, I may give you a question on sequences, but the question will not require you to recall some fact I presented in class. Rather, I will give you the facts needed to do the problem and test you on your ability to use these facts and write up solutions. That is, the focus is on your ability to communicate and solve problems in general. With that being said, there will still be big problems presented that you will be responsible for like the previous exam.

2 Review

Before we continue with our studies, we will review cryptarithmetic since that homework was turned in. I saw that a common issue was with the "CAT" problem. So I will go over the solution for this problem:

Problem. *Find values for C, A, T, D so that the following makes sense:*

$$\begin{array}{cccc} & C & A & T \\ & C & A & T \\ & C & A & T \\ + & C & A & T \\ \hline & D & C & A \end{array}$$

Solution.

1. I'm sure most of you tried $C = 1$ and found that it got you no where–though I think one person use $C = 1$ to solve the problem. In any case, remember the point is to *Guess and Check* here and if you see that $C = 1$ does not work, you should try $C = 2$. Note no other value works since the sum is three digits.

2. Guess $C = 2$. Then we have

$$\begin{array}{cccc} & 2 & \text{A} & \text{T} \\ & 2 & \text{A} & \text{T} \\ & 2 & \text{A} & \text{T} \\ + & 2 & \text{A} & \text{T} \\ \hline & \text{D} & 2 & \text{A} \end{array}$$

3. It follows that the guess $D = 8$ seems reasonable. So,

$$\begin{array}{cccc} & 2 & \text{A} & \text{T} \\ & 2 & \text{A} & \text{T} \\ & 2 & \text{A} & \text{T} \\ + & 2 & \text{A} & \text{T} \\ \hline & 8 & 2 & \text{A} \end{array}$$

4. Here we need to do some more guessing and checking. Since we have $D = 8$, we note that adding the A's cannot carry over a 2 or greater. Note further the only way to add the four A's to get a two is either have the T's carry something over or set $A = 8$. But $A = 8$ carries over a 3, so that won't work. Thus we try to get the T's to carry over something.

5. Guess $A = 0$. Then,

$$\begin{array}{cccc} & 2 & 0 & \text{T} \\ & 2 & 0 & \text{T} \\ & 2 & 0 & \text{T} \\ + & 2 & 0 & \text{T} \\ \hline & 8 & 2 & 0 \end{array}$$

Now what can T be? We find that in fact $T = 5$ works since $T + T + T + T = 20$, which gives us the zero for A and the two for C.

Therefore, we have the following works

$$\begin{array}{cccc} & 2 & 0 & 5 \\ & 2 & 0 & 5 \\ & 2 & 0 & 5 \\ + & 2 & 0 & 5 \\ \hline & 8 & 2 & 0 \end{array}$$

3 Some Pure Math Basics

Before we get started, I wanted to provide you with some terminology that is always used in mathematics so that there is no confusion. Mathematicians spend most of their time studying three different things: definitions, theorems, and proofs. In fact, one can say most of math is just seeing how definitions, theorems, and proofs interact and connect logically. We all know what a definition is. But what is a theorem to a mathematician?

Definition 1 (Theorem)**.** *A mathematical theorem is a mathematical statement that has been shown to be true by logic alone.*

One can think of theorems as basically mathematical facts. Let's first list some mathematically interesting theorems:

Theorem 1. *By stretching alone, you can reshape a doughnut shaped object into a coffeecup shaped object, but you cannot reshape a spherical shaped object into a coffeecup shaped object.*

This innocent looking theorem is actually a really important result in mathematics and takes around half a semester of senior level mathematics to prove.

Theorem 2. *Place a bunch of marbles on an infinitely large table. Then either all marbles line up perfectly on a line, or there's a line that contains exactly two points.*

To convince yourself this is true, draw a few points on a piece of paper and test this claim.

To demonstrate what we mean by a proof, let's now go over an example of a theorem we all know, or at least heard of:

Theorem 3 (Pythagorean Theorem)**.** *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*

How do we know this is true? This is where the use of a mathematical proof comes in. However, before we show the proof of the Pythagorean Theorem, we should define loosely what a mathematical proof is.

Definition 2 (Proof)**.** *A sequence of logical steps that allows the reader to believe without a doubt the given theorem is in fact true.*

So let's look at the proof of the Pythagorean Theorem using geometry.

Proof. Refer to in-class drawings or look it up. □

4 Sequences

We will continue our studies on sequences and related problems. First we will try to gain a stronger understanding and intuition on sequences. Then we will review the Fibonacci sequence and discuss some of its cool properties.

4.1 Review and Examples

Here we will consider some examples of sequences to get used to the notational tools used for sequences. We often want to write sequences in some short hand notation because writing out, say, 10 numbers of the sequence can be a bit tedious. So we will see a nice way to represent and even construct sequences by using a compact equation.

Example. Let $a_n = 3n$, where n is a positive whole number. For short hand, we write (a_n) to represent the sequence. So what are the numbers of the sequence (a_n)? We simply just plug in values for n and list the numbers:

n	a_n
1	3
2	6
3	9
4	12
5	15
6	18
7	21
8	24

We thus find that (a_n) is the sequence $3, 6, 9, 12, 15, 18, 21, 24, \ldots$. That is, (a_n) is the sequence of multiples of 3.

Let's now consider a bit more complicated sequence.

Example. Let $a_n = n^2 + 1$, where n is a positive whole number. We would like to find what (a_n) is. Let's make a table and just plug in values for n.

n	a_n
1	2
2	5
3	10
4	17
5	26
6	37
7	50
8	65

We thus find that (a_n) is the sequence $2, 5, 10, 17, 26, 37, 50, 65, \ldots$.

We see that writing an equation for the nth term a_n is a nice way of representing a sequence without having to write out every term. That is, for example, we can look at the sequence in the first example as the sequence (a_n) that satisfies $a_n = 3n$.

4.2 Successive differences

In this section we will see what it means to take the successive differences of a given sequence. This is not always particularly useful, but it can sometimes help gain an intuition on how a sequences acts and how two sequences may differ. We will see that we can in fact create a new sequence by taking the successive differences of a given sequence. Sometimes this shows a really interesting property about the sequence, as you will see in the homework. So let's start out with a definition:

Definition 3 (Successive difference)**.** *Let (a_n) be a sequence of numbers. Then the nth successive difference of (a_n) is the difference $a_{n+1} - a_n$. For example, the first successive difference is $a_2 - a_1$; the second is $a_3 - a_2$; and so on.*

We shall now see this definition in action with a simple sequence.

Example 1. Let (a_n) be the sequence defined by $a_n = n$; i.e., this is the sequence $\mathbb{N}$ from before. Construct a new sequence using the successive differences of the sequence (a_n)

Solution 1. Like usual, we construct a table to organize our information. For convenience, we will first write out the sequence:

n	a_n
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8

From here, we can construct a table of successive differences:

n	$a_{n+1} - a_n$
1	2-1=1
2	3-2=1
3	4-3=1
4	5-4=1
5	6-5=1
6	7-6=1
7	8-7=1
8	9-8=1

We see that we can construct a new sequence (b_n) by setting $b_1 = a_2 - a_1 = 1$, $b_2 = a_3 - a_2$, and so on; i.e., $b_n = a_{n+1} - a_n = 1$. Therefore the sequence created by the successive differences of (a_n) simply the sequence of 1's.

Example 2. Let (a_n) be the sequence defined by $a_n = n^2$; i.e., this is the sequence of square numbers. Construct a new sequence using the successive differences of the sequence (a_n).

Solution 2. Again, for convenience, let's construct a table for the sequence (a_n).

n	a_n
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64

From here we can construct a table of the successive differences of (a_n):

n	$a_{n+1} - a_n$
1	3
2	5
3	7
4	9
5	11
6	13
7	15
8	17

We see that this is a familiar sequence. The successive differences of the sequence (a_n) gives us the positive odd numbers (excluding 1).

Let's do one more.

Example 3. Let (a_n) be a sequence defined by $a_n = \frac{n(n+1)}{2}$; i.e., the triangle numbers. Construct a new sequence using the successive differences of the sequence (a_n).

Solution 3. Again, for convenience, let's construct a table for the sequence (a_n).

n	a_n
1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36

From here we can construct a table of the successive differences of (a_n):

n	$a_{n+1} - a_n$
1	2
2	3
3	4
4	5
5	6
6	7
7	8
8	9

We see that this is a familiar sequence. The successive differences of the sequence (a_n) gives us the positive whole numbers (excluding 1).

4.3 Fibonacci's Return

4.3.1 Fibonacci's sequence in a nice equation

Here we will review the Fibonacci sequence and see some of its (amazingly cool) properties. Recall the Fibonacci sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. If we let F_n be the nth number in the Fibonacci sequence and (F_n) represent this sequence as we did above, we have that $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$, which is clearly a difficult way of representing (F_n). There is, however, a nice way to represent the Fibonacci sequence using an equation for the nth term F_n.

Recall that the nth number in the Fibonacci sequence is found by adding the previous two numbers together; i.e., the nth Fibonacci number, F_n, is found by adding F_{n-1} to F_{n-2}. That is, $F_n = F_{n-1} + F_{n-2}$. We thus see that we can also write the Fibonacci sequence in a nice compact form. As an example, I will give you a theorem that is about the Fibonacci numbers and how Fibonacci numbers can be used to express any positive whole number.

4.3.2 Neat Theorem about Fibonacci's Sequence

Theorem 4 (Zeckendorf's Theorem)**.** *Every positive whole number can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers.*

Let's first see what exactly is this theorem stating. Basically, it states that if you give me a positive whole number x, I can find two non-consecutive numbers in the Fibonacci sequence such that we can write x as a sum of these two Fibonacci numbers. Consider the following examples: 10 can be written as $2+8$; 45 can be written as $34+11$; 99 can be written as $89+8+2$; 200 can be written as $144+55+1$; and so on.

4.3.3 Application of Fibonacci's Sequence

As I have mentioned, Fibonacci's sequence shows up everywhere. Let's consider some places where the sequence shows up in really random places.

Example. Fibonacci's sequence can actually convert kilometers to miles in a really cool way that uses Zeckendorf's theorem. However, we must restrict our conversions from whole number kilometers to an approximation of miles. Recall 1km is approximately 0.6213miles. Using a calculator, we find that 8km is about 5miles; 13km is about 8miles; 21 km is about 13miles; 34km is about 21 miles; and so on. Notice that this is just the Fibonacci sequence in action! Note that we don't really have many restrictions here because by Zeckendorf's theorem, we can write any positive whole number as a sum of Fibonacci numbers. For example, suppose we wanted to convert 18km to miles. Well 18km=13km+5km=8miles+3miles=11miles. Let's do a few more: 26km=21km+5km=13miles+3miles=16miles; 39km=34km+5km=21miles+3miles=24miles. However, we do find an error in the following case: 25km=21km+3km+1km. However, we know 1km is about 0.5miles, so we can say 25km=21km+3km+1km=13+2miles+.5miles=15.5miles.

So why is this happening? I'll leave you with this. Note that 1km is about 0.6213miles. Note further that $1/\phi$ is about 0.618, where ϕ is the golden ratio: $\phi = \frac{1+\sqrt{5}}{2}$. What do you think is going on here?

5 Pause and Reflect

We will now change gears quite a bit and attack a variety of problems while focusing on the strategy of *Pause and Reflect*. It is often the case that when we are trying to solve a problem, there are three natural stages of *In the Beginning*, *In the Middle*, and *In the End*. Generally, the beginning is the point where we start assessing the problem and what exactly is the given problem asking. Next, the middle of a problem is during the time of implementing whatever solution attempt you have came up with. By the end of the problem we mean the point just after you have solved the problem.

Let us go through each of these parts of a problem while considering several problems.

5.1 In the Beginning

Things to consider while starting out on a problem:

1. Reflect first how the problem is stated.
2. Do you understand it?
3. What is the "unknown".
4. What are the conditions of the problem?
5. Do you have enough information to solve the problem?
6. What should the answer look like?
7. Can you recall a similar problem?
8. Can you *See it* more clearly?
9. Can you *Simplify it*?
10. Can you now develop an effective plan?

With these things to consider in mind, we will go over three problems.

5.2 In the Middle

A very useful skill when problem solving is being able to *Monitor progress*. That is

1. Are you stuck?
2. Have you made any progress?
3. What methods have you tried and failed?
4. What mistakes have been made?

When you ask yourself these sorts of things, sometimes you may make the problem easier on yourself. You may prevent yourself from going down incorrect routes in the future or perhaps pave a path you can use for future similar problems.

It is sometimes the case you get completely stuck on a problem and have absolutely no idea how to continue. At this point, it is sometimes best to put the problem aside for a given time. You may want to sleep on it, or go for a walk, or stop and listen to some music. I do this quite frequently for my own homework problems. Sometimes a single problem will take me somewhere between 5 and 10 hours to complete–during that time, I find it refreshing and helpful to walk around campus and clear my mind. Other times I will switch what I am doing to something else, like playing chess online or writing music. This can help refresh your brain and help you start anew on your problem.

One last helpful tip is to do the following. Grab someone really, really patient or a friend from class and explain the problem to them. Explain what the problem is asking and how you think you might solve it. Doing this has solved so many of my problems it's almost annoying, but I strongly suggest attempting this if you can.

5.3 In the End

Okay, so now you've solved the problem. What you should not do is forget everything that has to do with this problem. You should try to learn as much as you can from the problem you just did. Not only will this help you in solving future problems, related or not, there are some problems that are really cool and worthy of being shared. So here's a cheesy anecdote. So both my fiancée and I read math on the side and oftentimes when either of us come across some really interesting problem or mathematical fact, we often share and discuss what it is. At the very least, practicing something like this furthers your knowledge of the world, which is ultimately why you are here anyways.

Here are some other things to consider:

1. Where could you have been more efficient?
2. What have you learned that might help you solve problems in the future?
3. How could you extend your solution to more complicated problems?
4. Can you solve the problems using a different approach?
5. If you change the conditions in the problem, can you solve the new version?

5.4 Problems

So with all this in mind, we will consider several problems. Some of these problems will have solutions that might go against your intuition. The point is to see that sometimes you need to shift your way of thinking and understanding. So our goal is to learn from these following problems instead of necessarily learning how to do them; i.e., these problems are going to be used to help us become better problem solvers.

Problem 1 (Two Male Children)**.** *Mr. Smith has two children, at least one of whom is a male. What is the probability both children are males?*

Solution 1. Your initial guess based on intuition will probably guess that the probability is 50%. However, we need to ignore this initial intuition–it's actually incorrect. We need to shift our thinking and consider all the possible cases: since we know at least one of Mr. Smith's children is a male, we have the possible combinations: (male, male), (male, female), or (female, male). The outcome we are looking for is the (male, male) scenario and so our probability is 1/3; i.e., we have a $33.33\ldots\%$ chance of Mr. Smith having both children male.

Problem 2 (Two Male Children (modified))**.** *Mr. Smith has two children, the eldest one is male. What is the probability both children are male?*

Solution 2. We essentially solve the problem as we did in the previous version. Let's just list out the possibilities. By writing (male, female), we mean the younger child is a male and the older child is a female. So, using this convention, we only have two cases to consider since we know the eldest child is a male: (male, male) or (female, male). Note (male, female) is not allowed since this would require the eldest child to be female. Therefore, the probability that both children are males is 1/2; i.e., we have a 50% chance of Mr. Smith having both children male.

Let's do another problem that challenges our intuition.

Problem 3. *A piece of string is cut so that it wraps exactly once around the circumference of a basketball. The string is then lengthened by 5 feet and made into a circle with the basketball at the center. Now imagine a string wrapped around the equator of the earth, lengthened by 5 feet and reformed into a circle with the earth in the center. Is the string farther away from the basket ball or from the earth?*

Solution 3. Like the previous problems, our intuition might guide us the wrong way. Depending on how you look at it, you might convince yourself the earth would be farther way since the earth is so large. You might also convince yourself the basketball is farther since it is so much smaller. So let us *Pause and Reflect.*

1. What are some strategies we know that might help?
   Well we can try *Seeing it* by drawing pictures.

2. Let's ask ourselves, What is the unknown in this problem?
   Well we are dealing with distances so it must be look into the distance from the central object to the new rope. Let's add this to the image.

3. What are some equations we can use?
   We have that the circumference C of a circle of radius R is $C = 2\pi R$.

4. We note that if we find the radius of the new rope circle and subtract from it by the radius of the initial rope circle, we may find the distance from the central object to the new rope circle.

5. Let C be the circumference of the initial rope circle and R its radius. Then $R = \frac{C}{2\pi}$.

6. Note that the circumference of the new rope circle is just $C + 5$ since we have increased its length by 5.
   Therefore the radius of the new rope circle is $\frac{C+5}{2\pi} = \frac{C}{2\pi} + \frac{5}{2\pi}$.

7. We find that for either case, the distance from the central object to the new rope circle is $\frac{C}{2\pi} + \frac{5}{2\pi} - R = \frac{5}{2\pi}$.

8. Therefore, the distance in either case is $\frac{5}{2\pi}$; i.e., both the basketball and the earth are the same distance from their respective new rope circles.

The next problem is so counterintuitive it's worth keeping in mind when doing seemingly simple problems.

Problem 4.