1 Magic Squares

1.1 Introduction

Today we will learn about the mathematical object called a *Magic square*.

Definition 1 (Magic Square)**.** *A magic square is an arrangement of distinct whole numbers in a square grid, where the numbers in each row, and each column, and the numbers in the main diagonal, all add up to the same magic number. We say a magic square is an n by n magic square if the underlying square grid n by n.*

Definition 2 (Magic Number)**.** *The magic number is the number each of the columns, rows, and diagonals of a magic square sum up to.*

Example 1. Does the following satisfy the conditions in the definition of a magic square:

2	7	6
9	5	1
4	3	8

Solution 1. We will check this is a magic square directly.

1. Let's compute the sums of the numbers in the rows, columns, and main diagonals.
2. By diagonal, I mean the diagonals containing 2,5,8 and 4,5,6, and not the diagonals 7,9, or 7,1, and so on.
3. Sum the rows: 2+7+6=15; 9+5+1=15; 4+3+8=15.
   Looking good.
4. Sum the columns: 2+9+4=15; 7+5+3=15; 6+1+8=15.
5. Sum the diagonals: 2+5+8=15; 4+5+6=15.
6. Conclusion: this is in fact a magic square.

So we see that for a 3 by 3 square grid, we can in fact construct a magic square.
Question: Will this always work?
Answer: Kind of, but no.
Consider the following example.

Example 2. Can we make a 2 by 2 magic square?

Solution 2. We begin by drawing an empty 2 by 2 square grid:

1. Ask ourselves, can we or can we not place the number $1, 2, 3$, or 4 in the squares to form a magic square? We will show this is impossible.
2. Use guess and check to come up with insight. Let's form the following attempt:

1	2
3	4

3. We see that the column 2,4 adds up to six, yet the column 1,3 adds up to 4. So this doesn't work.

4. We observe that we will always have a 1,3 diagonal, column, or row and a 2,4 diagonal, column or row.

5. Conclude, we will get a sum of 4 and a sum of 6. Therefore, there exists no 2 by 2 magic squares.

It turns out that all n by n magic squares exist so long as $n > 0$ and $n \neq 2$.

Example 3. What is the 1 by 1 magic square?

Solution 3. We see that the only magic square is

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So this concludes the introduction on magic squares. Next we will review the history of them and then the construction of all 3 by 3 magic squares.

1.2 History

So I will provide a little historical background because it's pretty interesting. Magic squares have been around longer than any of you probably realize. How long do you think they have been around?

1. The study of magic squares falls under the field of mathematics called recreational mathematics.
    - This field is usually practiced by amateur mathematicians and math enthusiasts, though professional mathematicians have studied it in detail.
2. Magic squares have been around since as early as 80AD.
    - There is Chinese literature that dates to this time refers to a tale that talked about a turtle emerging from a sea and on it's back was a 3 by 3 magic square.
    - There is also a 10th century 4 by 4 magic square constructed on an Indian temple.
3. Nowadays, there is a general study being done for "magic shapes" of different sizes.

This concludes the discussion on the historical background of magic squares. We now move on to the problem solving aspect of magic squares and how to construct them.

1.3 Construction

We will now discuss how to construct a 3 by 3 magic square. The method to do so primarily relies on the Stir it up strategy of guess and check.

So how do we go about constructing a magic square? In general, the answer to this question is very, very difficult. However, we can answer the question for 3 by 3 magic squares. So let's solve the following problem:

Problem 1. *Construct a 3 by 3 magic square.*

Answer 1. We will simplify the problem by using guessing and checking. We will let M mean the magic number.

1. Our first step is simply to draw a blank 3 by 3 grid:

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2. We ask ourselves, Can we find what M *must* be?

    (a) Let's fill the grid with variables:

a	b	c
d	e	f
g	h	i

.

(b) Let's sum up each row. By definition, $a+b+c=M$, $d+e+f=M$, and $g+h+i=M$. It follows that

$$a+b+c+d+e+f+g+h+i=3M.$$

(c) What is the sum of the numbers in the hypothetical square? Well we know that we must use each of the numbers $1,2,\ldots,9$ exactly once and so the sum is

$$1+2+3+4+5+6+7+8+9=9(10)/2=45=3M.$$

(d) Therefore, we find that $M=15$.

3. We can just take it as fact that $M=15$ if the previous math didn't make sense.

4. Next consider what the center square must be. To do so, guess and check.

5. Try 4. We see that we will always have a row or column or diagonal with a 1:

		x
	4	
1		

,

which requires that $x=10$ so that $1+4+x=M=15$. But we restricted x to be between 1 and 9. A similar argument shows 1, 2 nor 3 work.

6. Try 6. We will always have a row or column or diagonal with a 9:

		x
	6	
9		

,

which requires that $x=0$ so that $9+6+x=M=15$. But we restricted x to be between 1 and 9. A similar argument shows 7,8 nor 9 work.

7. Therefore, the center square must be 5.

8. The remainder of the solution is now just guess and checking.

Let's first define what it means for two magic squares to be symmetric.

Definition 3 (Symmetric Magic Squares)**.** *Two magic squares are symmetric if one can be rotated or reflected into the other.*

Example 4. Consider the magic square from before:

2	7	6
9	5	1
4	3	8

,

If we rotate this square by 90^o, to get:

4	9	2
9	5	7
8	1	6

,

Would we get a magic square?

Solution 4. Check for yourself that this is still a magic square.

Problem 2. *How many magic squares are symmetric to the magic square:*

2	*7*	*6*
9	*5*	*1*
4	*3*	*8*

Answer 2. Recall that the number of symmetries of a square was 8. Therefore, applying each of these symmetries will give us a new magic square and hence there are 8 magic squares symmetric to:

2	7	6
9	5	1
4	3	8

It is a matter of fact that if we consider two magic squares to be equal if they are symmetric, that there is only 1 unique 3 by 3 magic square. However, there are actually 880 4 by 4 magic squares. There is a pretty neat 4 by 4 magic square as the following example will show.

Example 5. Is the following a magic square:

12	6	15	1
13	3	10	8
2	16	5	11
7	9	4	14

Solution 5. There are a lot of sums we would have to compute. I will leave most of them as an exercise, but will compute a column, a row, and the diagonals so what a diagonal means is clear.

1. We first compute the magic number. Doing a similar process will show that $M = 34$
2. Let's sum a row: 12+6+15+1=34.
3. Let's sum a column: 12+13+2+7=34.
4. Now, by diagonal we mean the diagonals 12,3,5,14 and 7,16,10,1.
5. Sum the diagonals: 12+3+5+15=34; 7+16+10+1=34.
6. Do the rest on your own if you want.

So, we see that this is an example of a magic square. This one is special though because if we consider any 2 by 2 square grid within the magic square, the sum of the numbers is also 34. For example, consider the two 2 by 2's:

12	6
13	3

5	11
4	14

We find that 12+6+13+3=34 and 4+11+5+14=34.

2 *PSSSP*

Let's now analyze the work we went through in terms of the *PSSSP* strategies. We will go through each strategy and detail how the solution to this problem may have used them.

1. Be **P**roactive: This problem wasn't the sort we could solve immediately. We *had* to attack and keep attacking in order to solve it.

2. **S**implify it: We simplified the problem by first determining the magic number and then showing the center must contain the number 5.

3. **S**ee it: We had to physically draw out the magic squares to solve the problem.

4. **S**tir it up: This problem was all about guessing and checking.

5. **P**ause and Reflect: