Oct15 Lecture Notes

1 Sequences

The topic of study today will be a problem posed by a mathematician called Fibonacci. Before we get to the problem, we need to study the field of mathematics that studies a mathematical object called a numerical sequence. So we will acquire an understanding and intuition of what numerical sequences are and then consider the problem first posed by Fibonacci. We start off with a definition and then go over examples.

Definition 1. *A sequence of numbers is an ordered list of usually an infinite amount of numbers* $a_1, a_2, a_3, \cdots$. *We write* a_n *for the nth number; i.e., the first number is written* a_1*, the second is written* a_2*, and so on.*

1.1 Examples

To make sure we really understand this definition, we will go over a lot of examples.

Example. Let us consider the sequence we all know and love: the positive whole numbers. Here we have $a_1 = 1, a_2 = 2, a_3 = 3, \cdots$, and so on. That is, we can say $a_n = n$. So, if $n = 5$, what is a_n? It is $a_5 = 5$. If $n = 123$, what is a_n? It is $a_{123} = 123$. We see that there is a natural sort of ordering here: the first number is 1, the second is 2, and so on. We denote this sequence by $\mathbb{N}$.

Example. Let's consider another sequence we all know and love: the positive even numbers. Here we have $a_1 = 2, a_2 = 4, a_3 = 6, \cdots$, and so on. In fact, we can say $a_n = 2n$. We say the first number is 2, the second is 4, and so on. We may write this as $2\mathbb{N}$.

We will see that $2\mathbb{N}$ also represents another sequence other than just the positive even numbers. Before we say more, we must define something first.

Definition 2. *An* n*-gon is a* n *sided regular polygon.*

Example. A triangle is a 3-gon. A square is a 4-gon. A pentagon is a 5-gon.

Using this definition, we can construct $2\mathbb{N}$ in a different way.

Example. Let a_3 be the number of symmetries of a 3-gon; viz., the number of symmetries of a triangle. So $a_3 = 6$. Let a_4 be the number of symmetries of a 4-gon; viz., the number of symmetries of a square. So $a_4 = 8$. Let a_5 be the number of symmetries of a 5-gon; viz., the number of symmetries of a pentagon. So $a_5 = 10$. Continuing with this, we let a_n be the number of symmetries of a n-gon. Then $a_n = 2n$ because n-gon have $2n$ symmetries. We see that, if we ignore the fact that this sequence starts at a_3, this sequence is the same as $2\mathbb{N}$, the sequence of positive even whole numbers.

Let's now go over a sequence some of you might see what it represents immediately.

Example. Consider the sequence $3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, \cdots$. This is of course can be seen as the sequence for the digits of the number π.

Example. Consider the sequence $-1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, \cdots$. This can be seen as the sequence for $-1.33333\cdots$. So we see the numbers in a sequence may also repeat and be negative.

1.2 Constructing sequences

We will see we can construct sequences in very interesting ways. The next sequence we will consider is a sequence we have seen several times, but I will construct it using some geometry. This sequence of numbers is called the triangle numbers, but you have seen it as the sum of the first n whole numbers.

Example (Triangle numbers)**.** We start off with one ball. We let a_1 be the number of balls; i.e., $a_1 = 1$. Underneath the first ball, draw two balls to make a triangle. We let a_2 be the number of balls now; i.e., $a_2 = 3 = 2 + 1 = 2 + a_1$. Under these two balls, draw three balls to make a triangle. Like before, we let a_3 be the number of balls now; i.e.,$a_3 = 6 = 3+2+1 = 3+a_2$. Under these three balls, draw four balls to make a triangle. Like before, we let a_4 be the number of balls now; i.e., $a_4 = 10 = 4+3+2+1 = 4+a_4$. Continue doing this, we have at the "nth" stage, a_n is the total number of balls and can be written $a_n = n + a_{n-1}$.

Let us now construct similar sequence. This is another sequence you have already seen, but again, I will construct it using some geometry. This sequence of numbers is called the square numbers. The naming will agree with how you've seen it before because it is in fact the same sequence as the squares of positive whole numbers.

Example (Square numbers)**.** We start off with one square. We let a_1 be the number of squares; i.e., $a_1 = 1$. From this, we add the minimum number of squares to construct a new square; i.e., we add three squares. We let a_2 be the number of squares now; i.e., $a_2 = 4$. We do the same as before and add the minimum number of squares to construct a new square; i.e., we add five squares. We let a_3 be the number of squares now; i.e., $a_3 = 9$. Do the same as before; i.e., we add seven squares. Letting a_4 be the number of squares now, we get $a_4 = 16$. Continuing doing this, we have at the "nth" stage, a_n is the total number of squares and can be written $a_n = n^2$.

Let's construct this sequence another way that is interesting.

Example. We start off with one triangle. Let a_1 be the number of triangles; i.e., $a_1 = 1$. Now add three triangles to make a new triangle (think of the triforce). Let a_2 be the number of triangles; i.e., $a_2 = 4$. Add five more triangles to make a new triangle. Let a_3 be the number of triangles; i.e., $a_3 = 9$. Next we add seven triangles to make a new triangle. Let a_4 be the number of triangles; i.e., $a_4 = 16$. If we continue this, we have at the "nth" stage, a_n is the total number of triangles and can be written $a_n = n^2$, and so this sequence does agree with the square numbers.

1.3 Fibonacci sequence

We will now consider what is probably the most popular sequence. We will first see how the sequence was original determined and then some of its most interesting properties. The sequence is called the Fibonacci sequence and it was constructed by the mathematician called Fibonacci. The way he constructed the sequence was by considering a puzzle he posed. The puzzle was originally to model the population growth of rabbits, though I don't believe Fibonacci was really interested in population growth. I think he was just interested in the mathematics behind it. Like a lot of things in mathematics, this sequence was actually known some time before Fibonacci. It was at least known to Indian mathematicians around 700AD and probably before that. However, it wasn't rigorously studied until after Fibonacci discussed the sequence in his book *Liber Abaci* in 1202AD.

So we will first consider the problem as Fibonacci posed it, analyze it in terms of this class, and then see how the sequence is usually constructed.

1.3.1 Fibonacci's Puzzle

Problem 1. *A newly born pair of rabbits, one male and one female, are put into a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on. How many rabbit pairs will there be after n months?*

Solution 1. On the first month, we start out with just a single pair and so they mate. On the second month, the first female gives birth to a new pair, so now we have 2 pairs. Here the original pair mates, but the new one does not. On the third month, the first female gives birth to a new pair and the first and first new pair mates. We now have 3 pairs. On the fourth month, the first females gives birth to a new pair, the first new pair gives birth to their first pair. Now we have five pairs.

Month	Number of pairs	Rabbits born this month
0	1	0
1	1	0
2	2	2
3	3	2
4	5	4

Note that doing this process will allow us to solve the problem when n is given; i.e., if the problem asked "How many rabbit pairs will there be after 10 months, this process will give us the solution exactly. However, to find the "nth" Fibonacci number is quite a bit more difficult. It can be done, but for this class we will leave that alone.

1.3.2 Another construction Fibonacci's sequence

Confer in class drawing or use Google to find construction using squares.

2 Application of sequences

2.1 Ulam's Spiral

So the sequence we will consider here is basically just the sequence $\mathbb{N}$ we talked about above. We will see how when $\mathbb{N}$ is written in a specific way, a quite beautiful pattern that has to do with primes emerges. However, before we get started, Ulam's spiral has a pretty entertaining back story I would like to share. So one way mathematicians communicate their ideas and work to other mathematicians is by holding colloquia and seminars. Basically, a mathematician writes a paper and then discusses the main results of the paper with a group of other mathematicians. So during a colloquium, a mathematician called Ulam found the presentation to be about a "long and very boring paper". As a result, he began doodling on some paper and came up with a way of writing $\mathbb{N}$ in such a way to show an interesting and quite beautiful pattern of the prime numbers. (Confer picture of Ulam's spiral on google.) What we find is that the prime numbers seemed to line up on diagonals. Why this is so surprising and interesting because it is extremely difficult to generate prime numbers in general and here we have a way that seems to generate primes in a nice way. It turns out that this way of generating primes is still not nice enough but certainly demonstrates a certain beauty in mathematics. I would like to mentioned lastly that, to make sure this pattern is interesting, if we were to plot numbers randomly in a similar square grid, we would not see such a pattern. That is to say, this pattern is almost unique to Ulam's spiral.

2.2 The Josh Sequence

With the Ulam's spiral in mind, I created a sequence of numbers that produces a nice pattern for even and odd numbers. The way this came about was after some time learning about the Ulam's spiral, my fiancée and I created a game whereby we would create sequences of numbers and have each other guess how the sequence was constructed. So I will construct the sequence geometrically, similar to the triangle and square numbers, and then do something similar to what Ulam did.

To construct the sequence, we do the following. Draw a single square and write 0 in it. We will now add squares in a spiral fashion as follows. We draw another square (of the same size) to the right. We ask ourselves, how many squares touch this new square? We find only one. We add this number to the number in the previous square to get $0 + 1 = 1$ and write this number in the new square. Next, we draw another square to start the spiral; i.e., the new square goes above the square with a 1 in it. We count how many squares are touching this new square: the squares with 0 and 1 both touch it giving us two squares. Add this to the previous number to get $1 + 2 = 3$ and write this in the new square. Draw a new square to the left of the previous and do the same process giving us a 6.

Next, we will see the pattern by circling the even numbers. We see that we divide the square grid into four regions of approximate triangles.