# 1.3 Exponential Models

### Compound Interest

Principal $P$ invested for a period of time at an interest rate of $i$. Then the amount at the end of the first period is:

$$\text{principal} + \text{interest} = P + Pi = P(1+i)$$

**Example.** Annual interest for bank account is 6% and its period is a month. Then $i = \frac{0.06}{12} = 0.005$.

**Definition 1** (Compound Interest)**.** *Interest for several periods gives compound interest.*

**Example 1.** \$1000 deposited at 6% per year interest and compounded monthly, then

$$\begin{aligned}
\text{amount at end of month } 1 &= P(1+i) = 1000(1+0.005) = 1000(1.005)\\
\text{month } 2 &= 100(1.005)(1+i) = 100(1.005)^2\\
\text{month } 3 &= 1000(1.005)^2(1+i) = 1000(1.005)^3\\
\text{month } n &= 1000(1.005)^n
\end{aligned}$$

In general, if $F$ =future amount, $P$=initial amount that earns interest at a rate per period of $i$, then

$$F = P(1+i)^n$$

after $n$ periods.

**Example 2.** \$100 deposited into account with annual yield of 8% compounded quarterly. Find $F$ after 5 years.

**Solution 1.** $P = 1000$, $i = \frac{0.08}{4} = 0.02$, $F = 1000(1+0.02)^n$
Note 5 years = 20 quarters = 20 periods so after 5 years, $F = 1000(1.02)^{20}$

**Definition 2** (Compound Interest (expanded on))**.** *Suppose a principal $P$ earns interest at the annual rate of $r$, and interest is compounded $m$ times a year. Then $F$ after $t$ years is*

$$F = P(1+i)^n = P(1+\frac{r}{m})^{mt}$$

*where $n = mt$=number of time periods and $i = r/m$ =interest per period.*

**Example 3.** $P = 1000$ yields 9% annually. Find $F$ after fifth year if compounding is monthly.

**Solution 2.** $P = 1000$, $r = 0.09$, $m = 12$, $t = 5$. So $F = 1000(1+0.09/12)^{12\times 5}$.

### Present Value

Given $F = P(1+r/m)^{mt}$ and say we want to know how many dollars $P$ to set aside so we will have a future amount of $F$ dollars after $t$ years. We solve for $P$:

$$P = \frac{F}{(1+r/m)^{mt}} = \text{Present value}$$

**Example 4.** Want \$20,000 after 18 years. Can earn 9% quarterly. Find present value.

**Solution 3.** Here $r = 0.09$, $m = 4$, $t = 18$, $F = 20,000$. Then

$$P = \frac{20000}{(1+0.09/4)^{4\times 18}} = 4029.69$$

### 0.0.1 Continuous Compounding

If a principal $P$ earns interest at an annual rate of $r$, and interest is compounded continuously, then $F$ after $t$ years is $F = Pe^{rt}$.

**Example 5.** $P = 1000$, yielding 9% annually, compounded continuously. Find $F$ after 1 year.

$P = 1000$, $r = 0.09$, $t = 1$. Then

$$F = Pe^{rt} = 1000e^{0.09}$$

### 0.0.2 Present Value

Like above, present value for continuous compounding is given by $P = Fe^{-rt}$.

### 0.0.3 Solving Equations with exponents

Recall $x^0 = 1$ for any $x \neq 0$

**Example 6.** (a) $2^x = 8$

(b) $9^x = 27^{4x-10}$

(c) $\frac{1}{3^{x^2}} = 3^{-2x+1}$

**Solution 4.** (a) Rewrite 8 as $2^3$ and so we find $2^x = 2^3$ when $x = 3$.

(b) Rewrite $9^x$ as $3^{2x}$ and $27^{4x-10}$ as $3^{12x-30}$. Then divide through by $3^{2x}$ to get $1 = 3^{12x-10-2x} = 3^{10x-30}$, which is true when $10x - 30 = 0$, or $x = 3$.

(c) Multiply through by $3^{x^2}$ to get $1 = 3^{-2x+1+x^2}$, which is true when $x^2 - 2x + 1 = 0$; i.e., when $x = 1$.

## Logarithms

Here we define a function that basically reverses exponentiation. For example, if $5^x = 2$, how could solve for $x$? The answer is the logarithm.

**Definition 3** (Logarithm)**.** *Let $a > 0$ and $a \neq 1$. If $x > 0$, then the logarithm base $a$ of $x$, denoted $\log_a x$ is defined as follows:*

$$y = log_a x \text{ if and only if } x = a^y$$

**Definition 4** (Common Logarithm/Log Base 10)**.** *Let $a = 10$, then*

$$y = \log_{10} x = \log(x) \text{ if and only if } x = 10^y$$

**Definition 5** (Natural Log/Log Base $e$)**.** *Let $a = e$, then*

$$y = \log_e x = \ln(x) \text{ if and only if } x = e^y$$

**Example 7.** (a) if $5 = 2^y$, then $y = log_2 5$. We can look at it as "taking the log base 2" on both sides:

$$\log_2(5) = \log_2(2^y)$$

where the $\log_2$ cancels the exponentiation of 2 by $y$.

(b) If $2 = e^y$, then $y = \ln(2)$.

(c) If $17 = 10^y$, then $y = \log(17)$.

**Properties of Log**

1. $a^{\log_a x} = x$ if $x > 0$ (so exponentiation undoes the logarithm).
2. $\log_a a^x = x$ for all $x$ (so logarithm undoes exponentiation).
3. $\log_a xy = \log_a x + \log_a y$
4. $\log_a x/y = \log_a x - \log_a y$
5. $\log_a x^c = c \log_a x$
6. Let $a > 0$. If $k = \ln a$, then $a^x = e^{kx}$

**Theorem** (Change of Base)**.** $\log_a x = \frac{\log_b x}{\log_b a}$.

$D(x) = 123e^{-0.7x}$ find $x$ if $D(x)$=something