3.1 Limits

Definitions

Definition 1 (Left Handed Limit)**.** Suppose a function f is defined on at least the interval (c, a) where $c < a$ (by "at least" I mean the domain of f may be larger than this interval). Then if as x approaches a from the left $f(x)$ approaches some value L, we write

$$\lim_{x \to a^-} f(x) = L = \text{Left Handed Limit}$$

and say the left handed limit of f as x approaches a from the left is L.

Definition 2 (Right Handed Limit)**.** Suppose a function f is defined on at least the interval (a, b) where $a < b$. Then if as x approaches a from the right $f(x)$ approaches some value L, we write

$$\lim_{x \to a^+} f(x) = L = \text{Right Handed Limit}$$

and say the right handed limit of f as x approaches a from the right is L.

Definition 3 (Limit of a function)**.** If f meets both conditions of a Left Handed Limit and a Right Handed Limit and these limits agree, we say the limit of f as x approaches a is L and write $\lim_{x \to a} f(x) = L$. That is, if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x),$$

then we write

$$\lim_{x \to a} f(x) = L,$$

to mean L is the limit of f as x approaches a.

Remark: f need not be defined at the point a for any of the above limits to exist.

Definition 4 (Existence of a limit)**.** If a function f has a limit L as x approaches a, we say the limit of f exists at $x = a$. Equivalently, the limit of f at $x = a$ exists if the right handed and left handed limits exist and agree; i.e., if

(a) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist,

and

(b) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

We say the limit of f does not exist at a point a if either the left handed limit or the right handed limit do not exist or if they do not agree; i.e., if

(a) $\lim_{x \to a^-} f(x)$ DNE or $\lim_{x \to a^+} f(x)$ DNE,

or

(b) $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

Definition 5 (Continuity)**.** A function f is continuous at a point a if $\lim_{x \to a} f(x) = f(a)$; i.e., f is continuous at a point a if the limit of $f(x)$ as x approaches a agrees with $f(x)$ at $x = a$.

Note that this requires f to be defined at a; i.e., $\lim_{x \to a} f(x)$ may exist, yet f may not be continuous there.

Definition 6 (Discontinuity)**.** A function f is discontinuous at a point a if it is not continuous there.

Note a function is discontinuous where it is not defined (cf. $f(x) = 1/x$).

Concepts

Rules For Limits

1. If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.

   For what follows, assume that

$$\lim_{x\to a} f(x) = L \qquad \lim_{x\to a} g(x) = M$$

2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cL$.
3. $\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$.
4. $\lim_{x\to a}(f(x)g(x)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x)) = LM$.
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ if $\lim_{x\to a} g(x) = M \neq 0$.
6. $\lim_{x\to a}(f(x))^n = L^n$ for any real number n such that L^n is defined, $L \neq 0$ if needed.

Finding the Limit Graphically

cf. lecture.

Finding the Limit Using Cancellation

Example. Find

$$\lim_{x\to 1} \frac{x^2 - 2x + 1}{x - 1}.$$

Example. Determine if

$$\lim_{x\to 3} \frac{x^2 - 4x - 5}{x - 3}$$

exists.

Different Limits From Each Side

Example. Let f be a function defined by,

$$f(x) = \begin{cases} x & \text{if} \quad 0 \leq x \\ x + 1 & \text{if} \quad x > 0 \end{cases}$$

and compute $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

Example. Let f be the function defined by,

$$f(x) = \begin{cases} 1 - x & \text{if} \quad 0 \leq x \\ \sqrt{x^2 + 1} & \text{if} \quad x > 0 \end{cases}$$

and compute $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

Limit Does Not Exist if the Function is Unbounded

Example. Does

$$\lim_{x\to 0} \frac{1}{x}$$

exist?

Intervals of Continuity

Example. Determine the intervals for which the following function is continuous and write them as a union.

$$g(x) = \frac{x^2 - 1}{(x-1)(x+2)}$$

Points of Discontinuity

Determine the points for which the following function is discontinuous.

$$g(x) = \frac{x}{x^2 + x}$$

Finding Points of Discontinuity Graphically

cf. lecture.

Continuity of Polynomials and Rational Functions

1. Any polynomial function is continuous everywhere.

   Example. Let $p(x) = x^2 + 1$. Then note that $\lim_{x\to a} p(x) = a^2 + 1 = p(x)$ for all a. E.g., $\lim_{x\to 2} p(x) = 2^2 + 1 = 5 = p(2)$.

2. A rational function is continuous at every point the denominator is not zero.

   Example. The rational expression

   $$r(x) = \frac{x^2 - 2x + 1}{x - 1}$$

   is continuous for all x except $x = 1$ since at $x = 1$ the denominator is zero.

Necessary concepts to know

Here is a minimum list of concepts you should know.

1. The above concepts.
2. When does a limit exists?
3. When does a limit not exist?
4. How do you take the limits of polynomials and rational expressions?
5. What are the limit rules?
6. When is a function continuous at a point?
7. When is a function discontinuous at a point?

Worked Out Examples

Example.

1. Compute
   (a) $\lim_{x\to 1} \frac{x^2+2x+1}{x+1}$
   (b) $\lim_{x\to -1} \frac{x^2+2x+1}{x+1}$

(c) $\lim_{x\to -1} \frac{x^2-2x+1}{x+1}$

(d) $\lim_{x\to 2} \sqrt{x+1}$

(e) $\lim_{x\to 1}(x^2+2x+1)(x^3-1)$

(f) Let f be the function defined by,

$$f(x) = \begin{cases} 1-x & \text{if} \quad 0 \leq x \\ \sqrt{x^2+1} & \text{if} \quad x > 0 \end{cases}$$

and compute $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

2. Find where the following functions are discontinuous.

(a)