3.3 The Derivative

Definitions

Definition 1 (The Derivative)**.** If $y = f(x)$, the derivative of $f(x)$, denoted by $f'(x)$ is defined to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists.

Note: you might see y', $\frac{dy}{dx}$, and $\frac{d}{dx} f(x)$ to mean the derivative.

Remark: If a function has a derivative at $x = c$ we say the function is differentiable there.
Remark: Recall that the derivative at a point gives the slope of the function at that point.

Concepts

Using the limit definition of the derivative

Example. Let $f(x) = 5$. Find it's derivative.

Example. Let $f(x) = x^2 + x + 2$. Find the derivative $f'(x)$ at $x = 3$. Find the derivative of $f'(x)$ at any value of x.

Example. Let $g(x) = \frac{1}{x}$. Find the derivative $g'(x)$ at $x = 2$. Find the derivative of $g'(x)$ at any nonzero value x.

Example. Let $h(x) = \frac{1}{x+1}$. Find the derivative $h'(x)$ at $x = 0$. Find the derivative of $h'(x)$ at any value of x with $x \neq -1$.

Example. Let $k(x) = x^{3/2}$. Find the derivative $k'(x)$ at $x = 0$.

Finding the equation of the tangent line

Example. Let $f(x) = x^2 + x + 2$. Find the tangent line at $(2, f(2))$.

Example. Let $g(x) = \frac{1}{x}$. Find the tangent line at $(c, f(c))$.

Example. Let $h(x) = \frac{1}{x+1}$. Find the tangent line at $(0, f(0))$.

Example. Let $k(x) = x^{3/2}$. Find the tangent line at $(0, f(0))$.

Positive, zero, and negative derivatives

A derivative will either be positive, zero, or negative. We have the following situations:

1. Derivative is positive: positive slope, so function is increasing.
2. Derivative is negative: negative slope, so function is decreasing.
3. Derivative is zero: zero slope, so function is constant.

Differentiability implies continuity

Remark: If a function $f(x)$ is differentiable at a point $x = c$, it is continuous at $x = c$. That is, if

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists, then

$$\lim_{x \to c} f(x) = f(c).$$

Remark: **COMMON ERROR TO AVOID**: Continuity does **not** implies differentiability. A function may satisfy:

(i) f is continuous at $x = c$ and not differentiable at $x = c$
or

(ii) f is continuous at $x = c$ and is differentiable at $x = c$.

Note that this means if f is not continuous at a point, it is not differentiable at that point.

When derivatives fail to exist

The derivative of a function does not exist at x when

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

does not exist. There are at least two ways to see a function does not exist:

1. Graphically: cf. lecture.
2. Let $f(x) = 1/x$. Then f is not differentiable at $x = 0$; it isn't even defined here!

Derivatives and graphs

cf. lecture.

Necessary concepts to know

Here is a minimum list of concepts you should know.

1. Using the limit definition of the derivative to find $f'(x)$
2. Finding the equation of the tangent line.
3. Differentiability implies continuity.
4. When the derivative fails to exist
5. Using graphs to determine where functions do not have derivatives.
6. Using graphs to determine values of $f'(x)$.
7. What does the sign of the derivative tell you?