Elasticity of Demand

Definition (Elasticity of Demand)**.** If the demand equation is given by $x = f(p)$, then

$$E = -\frac{p}{x}\frac{dx}{dp}$$

is called the elasticity of demand.

1. If $E > 1$, then demand is elastic.
2. If $E < 1$, then demand is inelastic.
3. If $E = 1$, then demand has unit elasticity.

Example. Let $x = 1 - p^2$ be the price demand equation. Find $E(p)$ and determine if demand is elastic, inelastic, or unit elastic at the following values for p.

(a) $p = 2$

(b) $p = -2$

Solution. Solve for $E(p)$:

$$\frac{dx}{dp} = -2p$$

and so

$$\begin{aligned} E(p) &= -\frac{p}{x}\frac{dx}{dp} \\ &= -\frac{p}{1-p^2}(-2p) \\ &= \frac{2p^2}{1-p^2} \end{aligned}$$

.

Thus $E(2) = 2(2)^2(1-2^2) = -8/3$, and so demand is inelastic here. Note $E(-2) = E(2)$ and so demand is inelastic for $p = -2$ as well.

Theorem. *If $E < 1$, increasing price increases revenue. If $E > 1$, increasing price decreases revenue. If $E = 1$, revenue is optimized.*

Example. When is revenue optimized if the demand equation is given by $x = (1+p)^2$?

Solution. By the theorem, we need to find when $E = 1$, since this is when revenue is optimized. We find $\frac{dx}{dp} = -2 + 2p$. Thus,

$$E(p) = \frac{p}{x}\frac{dx}{dp} = -\frac{p}{(1+p)^2}(-2+2p).$$

We want $E(p) = 1$, and so, from here, we just solve for p, getting $p = 1/3$ is when our profit is optimal.

Note that $\frac{\Delta x}{x}100$ gives the percentage in change in demand and $\frac{\Delta p}{p}100$ gives the percentage in change in price. Thus, dividing these two numbers gives $\frac{p}{x}\frac{\Delta x}{\Delta y}$, which is approximately $-E(p)$. Thus, given $\frac{\Delta p}{p}100$ and $E(p)$, we may approximate $\Delta x/x$ by $-\frac{\Delta p}{p}E(p)$.

Next note that as price increases, demand decreases, and as price decreases, demand increases. That is to say, if change in price is negative, change in demand is positive, and if change in price is positive, change in demand is negative.

Example. Suppose that the price of some object is \$100 and $E(100) = .5$. Now suppose the price of the object is decreased by \$5. What is the approximate change in demand.

Solution. Since price is decreasing, we note from above that we know demand must be increasing. Next, we basically just need to apply the blurb from above. We have $E(100) = .5$ and $\frac{\Delta p}{p} = -5/100 = -0.05$, showing that we have a decrease in price by 5%. Thus, using the above approximation, we have an increase in demand by $5\% \times E(p) = 5\% \times 0.5 = 2.5\%$.

Example. Let $x = \sqrt{4-p}$ be the demand equation.

1. Find $E(p)$.
2. Find where the demand has unit elasticity.
3. Give interval for which demand is inelastic/elastic.
4. Find when revenue is maximum.
5. What is maximum revenue?

Example. Let $x = e^{p-p^2}$. Find $E(1)$.

Solution. We find $x' = (1-2p)e^{p-p^2}$. Then $x'(1) = -2$. Also, $x(1) = 1$. Thus, $E(1) = -\frac{1}{1} \times -2 = 2$. This tells us that the demand is going down by 2% per 1% in price at this price level.