Critical Points and Relative Extrema

In this section we examine the first derivative of a function and what it may reveal to us about the function itself. The first thing we consider is probably the most crucial to understand since subsequent material will rely heavily on it. That thing being **critical points** of some function which is defined as follows.

Definition. Let x be in the domain of some function f. If either $f'(x) = 0$ *or* f' does not exist at x (i.e., f is not differentiable at x), then x is called a critical point of f.

Remark. It is crucial for this definition that a critical point need by in the domain of f. We shall see that there are values x for which $f'(x)$ is not defined, yet x is not a critical point.

Example. Find the critical points of the following functions.

1. $f(x) = 1/x$
2. $g(x) = (x+1)^{1/2}$
3. $h(x) = \ln x$
4. $F(x) = x^5 - x^3 + 19$

Solution.

1. Its derivative is $f'(x) = -1/x^2$ and we see that $f'(0)$ is not defined. However, $f(0)$ is not defined either so $x = 0$ is not a critical point; i.e., 0 is *not* in the domain of f. Moreover, $f'(x)$ is continuous everywhere else and thus f has no critical points.
2. Its derivative is $g'(x) = \frac{1}{2}(x+1)^{-1/2}$. Note that neither g nor g' are defined on $(-\infty, -1)$ since here is where $x + 1 < 0$. However, $g'(0)$ is not defined, yet $g(0)$ is and therefore $x = 0$ is the only critical point of g.
3. Its derivative is $h'(x) = 1/x$. Note that h is only defined on $x > 0$ and so h' is also only defined by $x > 0$ and hence h has no critical points.
4. Its derivative is $F'(x) = 5x^4 - 3x^2$. We wish to find x such that $F'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3) = 0$. But this occurs when either $x = 0$ or $x = \pm\sqrt{\frac{3}{5}}$, as you can check by inputing these values or simply solving for x.

It might be surprising that the critical points of some function f tell us anything about the function f. However, given the critical points of a function and knowing where f' is continuous will allow us to determine where f is increasing or decreasing; viz., the first derivative may tell us information about the behavior of the function. Recall the following two facts,

1. If $f'(x) > 0$ on (a, b), then f is increasing on (a, b).
2. If $f'(x) < 0$ on (a, b), then f is decreasing on (a, b).

Therefore, if we may find intervals for which f' is positive or negative, we may find intervals for which f is increasing or decreasing, respectively. Thus the following theorem is useful.

Theorem. *Suppose f is a continuous function on some interval (a, b). Fix a point x in (a, b). If f never becomes zero on (a, b), then f has the same sign as $f(x)$ on (a, b).*

Remark. We are very well able to restate this theorem using f' in place of f and thereby set us up for determining where a function increases or decreases.

If you are curious where this theorem comes from, over some interval draw a graph strictly above the x-axis and observe that since the graph never crosses the x-axis (that is, the function never becomes zero on the interval), the sign of the function remains the same.

Example. Let $f(x) = 1 - x^2$. Note that f is certainly continuous and only has the two zeros at $x = \pm 1$. That is, f does not vanish anywhere on $(-1, 1)$. Moreover, $f(0) = 1 > 0$. Thus, by the previous theorem, given that f is continuous and non-vanishing on $(-1, 1)$, and $f(0) > 0$, we may conclude that $f(x) > 0$ for all $x \in (-1, 1)$.

We then have a nice way of determining when a function is increasing or decreasing by using this theorem.

Example. Let the derivative of a function g be given by $g'(x) = \frac{x-1}{(x-2)^3}$ for all $x \neq 2$. We wish to find where g is increasing or decreasing using the above theorem.

Note that g' is zero only at $x = 1$ and g' is not continuous only at $x = 2$. Thus, we need only check the sign of g' at a single points on each the intervals $(-\infty, 1), (1, 2)$, and $(2, \infty)$.

Note $0 \in (-\infty, 1)$, from which we find $g'(0) = -1/-2 = 1/2 > 0$, and so $g'(x) > 0$ on $(-\infty, 1)$. Next note that $3/2 \in (1, 2)$, from which we find $g'(3/2) = \frac{1/2}{(-1/2)^3} < 0$, and so $g'(x) < 0$ on $(1, 2)$. Lastly, by testing $3 \in (2, \infty)$, we find $g'(2) > 0$. Thus, g is increasing on $(-\infty, 1) \cup (2, \infty)$, since $g'(x) > 0$ here, and g is decreasing on $(1, 2)$, since $g'(x) < 0$ here.

We next consider where the function is decreasing. Similar to above, this occurs when $g'(x) < 0$, and hence it is sufficient to check where either $x - 1 > 0$ and $x - 2 < 0$, or when $x - 1 < 0$ and $x - 2 > 0$, since $\frac{a}{b} < 0$ when a and b have different signs. Thus, $g'(x) < 0$ when x in $(1, \infty) \cap (-\infty, 2) = (1, 2)$ or when x in $(-\infty, 1) \cap (2, \infty) = \emptyset$; i.e., $g'(x) < 0$ only when x in $(1, 2)$.

Example. Let $f(x) = xe^x$. We shall determine where f is increasing and where it is decreasing. We first note that $f'(x) = e^x + xe^x = 0$ on when $xe^x = -e^x$ and hence when $x = -1$. Thus, in determining where f is increasing and where it is decreasing, we need only check two values, one in $(-\infty, -1)$ and one in $(-1, \infty)$.

Now, certainly $-2 \in (-\infty, -1)$, and we check $f'(-2) = e^{-2} - 2e^{-2} = -e^{-2} < 0$, by $e^{-2} > 0$. Next, $0 \in (-1, \infty)$, and we check $f'(0) = e^0 + 0e^0 = 1 > 0$. Therefore, we conclude $f'(x) > 0$ on $(-1, \infty)$ and $f'(x) < 0$ on $(-\infty, -1)$, and hence f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

When we are given the function $f(x) = -x^2 + 1$, we can convince ourselves that the maximum of this function is $f(0) = 1$, but f has no minimum. Similarly, the minimum of $g(x) = x^2 - 1$ is $g(0) = -1$, but g has no maximum. However, we may face a function that has neither such maxima nor minima–consider $h(x) = x^3 - 1$, for example. We would like to generalize this notion to capture the notion of maxima and minima *locally*, where we shall specify what we mean by *local* soon. For example, if you graphed h, we find h has a *local* minimum at $x = -1$ and a *local* maximum at $x = 1$; i.e., if we focus around $x = -1$ or $x = 1$ in a small interval, h appears to have a minimum or maximum, respectively.

Now, by *local* we mean something is true within some small interval of a point. So, for example, if we say x is a local minimum of f, we mean that in some interval (a, b) of x, f is minimum at x in this interval. Next, instead of saying *local* minimum and maximum, we shall say **relative minimum and maximum**, respectively. Furthermore, we shall say **relative extrema** when speaking of *either* relative minima or maxima. We summarize these ideas as follows

If $f(x) \geq f(c)$ for all x in (a, b) for some $a < b$, then $f(c)$ is said to be a relative minimum of f.

If $f(x) \leq f(c)$ for all x in (a, b) for some $a < b$, then $f(c)$ is said to be a relative maximum of f.

It turns out that it is sufficient, in determining relative extrema, to firstly find all the critical points of a given function and secondly determine if the function is extremal at these given points. Thus we may use the first derivative to reveal information about the extremal behavior of a function. We thus state the celebrated **First Derivative Test**.

Theorem 1 (The First Derivative Test)**.** *Let c be an critical point. Let (a, b) be an interval of c not containing any other critical point. Choose u, v in (a, b) such that $a < u < c < v < b$. Then,*

If $f'(u) > 0$ and $f'(v) < 0$, $f(c)$ is a relative maximum of f

If $f'(u) < 0$ and $f'(v) > 0$, $f(c)$ is a relative minimum of f

If $f'(u)$ and $f'(v)$ have the same sign, then $f(c)$ is not a relative extremum

We demonstrate this by example first.

Example. Let $f(x) = x^2$. Note that $f'(x) = 2x$ and so f has only a single extremal point, namely $x = 0$. Next we apply the first derivative test. We find that $f'(-1) = -2 < 0$ and $f'(1) = 2 > 0$ and so $x = 0$ is a relative minimum.

However, we may show this another way. If $f(a) > f(0)$ and $f(0) > f(b)$ for all $a < 0 < b$, then f will have a relative minimum at $x = 0$. We find that $f(2) = f(-2) = 4 > 0$ and so $f(0) = 0$ is a relative minimum.

As a corollary, if $g(x) = -x^2$, we get $g(2) = g(-2) = -4$ and so g has a relative maximum at $x = 0$. This demonstrates the fact that if x is a relative minimum (maximum) for g, then x is a relative maximum (minimum) for $-g$.

Let's face a more difficult example.

Example. Let $g(x) = \frac{x+1}{(x-2)^4}$ and let's determine the relative extrema of g. We find $g'(x) = (x-2)^{-4} - 4(x+1)(x-2)^{-5}$, by the product rule. Thus, we want to find x such that $g'(x) = 0$. We find

$$(x-2)^{-4} - 4(x+1)(x-2)^{-5} = 0$$

and so, by multiplication of $(x-2)^5$,

$$(x-2) - 4(x+1) = 0,$$

so that $x = -2$ is a critical point.

Next we apply the first derivative test. We find that $f'(0) = 3/16$ and $f'(-4) = -1/1296$.

We compute $g(-2) = -1/(-4)^4 = -1/256$.

Schemes for problems

Finding critical points of a function

Let f be the function of interest. Then to find the critical points of f, we do the following.

1. Make note of the domain of f (e.g., where does f make sense).
2. Compute f'.
3. Make note of where $f'(x)$ is undefined.
4. Determine where $f'(x) = 0$.
5. If x is such that $f'(x)$ is undefined or $f'(x) = 0$, and if x is in the domain of f, then x is a critical point. Otherwise, ignore it.

Finding where a function is decreasing/increasing

Suppose you are given f or f'.

1. Compute f' (skip this step if already given f').
2. Find where f' is not defined, discontinuous, or equal to 0. Suppose for sake of simplicity that points a, b, and c are such values and $a < b < c$.
3. Construct the largest intervals for which f is continuous, defined, and nonzero. For example, for a, b, c above, we would choose $(-\infty, a), (a, b), (b, c)$, and (c, ∞).
4. Choose a nice point in each of the intervals and test the sign of f'.
5. Suppose I is one of the intervals and x_0 is the nice point you have chosen in this interval. If $f'(x_0) > 0$, then f is increasing on I and if $f'(x_0) < 0$, then f is decreasing on I.

Finding relative extrema

Suppose you are given f.

1. Find the critical points of f. Let x_0 be such a critical point.
2. Choose two points u and v such that $u < x_0 < v$ and there are no other critical points of f between u and x_0 nor between x_0 and v.
3. Find the signs of $f'(u)$ and $f'(v)$ and then apply the first derivative test.

Examples of the Schemes

Finding critical points of a function

Example. Find the critical points of the following functions.

(a) Let $f(x) = x^{1/3}$.

(b) Let $g(x) = e^x$.

(c) Let $h(x) = x^3 + x^2 + x + 1$.

Solution.

(a)
1. f is defined everywhere.
2. $f'(x) = \frac{1}{3}x^{-2/3}$ by the chain rule.
3. $f'(x)$ is only undefined when $x = 0$ (note this is when the tangent line of f would be vertical).
4. $f'(x)$ is never zero.
5. Therefore $x = 0$ is the only critical point of f.