Additional notes for 5.3/5.4

Limits at negative infinity

The following isn't necessary but is useful. We have the following identity

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(-x).$$

Example 1. 1.

$$\begin{aligned}\lim_{x\to-\infty} x^3 + x &= \lim_{x\to\infty}(-x)^3 + (-x) = \lim_{x\to\infty} -x^3 - x\\ &= \lim_{x\to\infty} -x^3(1+\frac{1}{x^2}) = \lim_{x\to\infty}(-x^3)\lim_{x\to\infty}(1+\frac{1}{x^2})\\ &= -\infty\cdot 1 = -\infty\end{aligned}$$

2.

$$\lim_{x\to-\infty} 1 + e^x = \lim_{x\to\infty} 1 + e^{-x} = \lim_{x\to\infty} 1 + \lim_{x\to\infty} e^{-x} = 1.$$

Limits involving exponentials

Recall that the exponential function can be treated as the largest degree polynomial, even though it is not a polynomial. So when we have

$$\lim_{x\to\infty}\frac{1+x}{x^2+e^x},$$

we can pretend that e^x is the largest degree term in the denominator and proceed as usual

$$\begin{aligned}\lim_{x\to\infty}\frac{1+x}{x^2+e^x} &= \lim_{x\to\infty}\frac{e^x(\frac{1}{e^x}+\frac{x}{e^x})}{e^x(\frac{x^2}{e^x}+1)}\\ &= \lim_{x\to\infty}\frac{\frac{1}{e^x}+\frac{x}{e^x}}{\frac{x^2}{e^x}+1}\\ &= \frac{\lim_{x\to\infty}\left(\frac{1}{e^x}+\frac{x}{e^x}\right)}{\lim_{x\to\infty}\left(\frac{x^2}{e^x}+1\right)}\\ &= \frac{0}{1} = 0,\end{aligned}$$

where we have used the fact that $\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{x^2}{e^x} = 0$.

Let's see another example. Suppose we have

$$\lim_{x\to-\infty}\frac{1+e^x}{1+2e^x}.$$

We now both have a limit at negative infinity and exponentials. We find do the trick from above.

$$\lim_{x\to-\infty}\frac{1+e^x}{1+2e^x} = \lim_{x\to\infty}\frac{1+e^{-x}}{1+2e^{-x}}.$$

Now, we needn't factor as we can see

$$\begin{aligned}\lim_{x\to\infty}\frac{1+e^{-x}}{1+2e^{-x}} &= \frac{\lim_{x\to\infty}(1+e^{-x})}{\lim_{x\to\infty}(1+2e^{-x})}\\ &= \frac{\lim_{x\to\infty}1+\lim_{x\to\infty}e^{-x}}{\lim_{x\to\infty}1+\lim_{x\to\infty}e^{-x}} = \frac{1}{1} = 1,\end{aligned}$$

where we have used the fact that $\lim_{x\to\infty} e^{-x} = \lim_{x\to\infty}\frac{1}{e^x} = 0$.

Finding Horizontal Asymptotes

Say you are asked to find the equation of the line of an horizontal asymptote. This is how you would do this.

Example. Let $f(x) = \frac{1+2x}{2+3x}$ and find its horizontal asymptotes.

To do this, we must determine the behavior of f at $\pm\infty$. We first find that

$$\lim_{x\to\infty} \frac{1+2x}{2+3x} = \lim_{x\to\infty} \frac{x(\frac{1}{x}+2)}{x(\frac{2}{x}+3)} = \frac{\lim_{x\to\infty}(\frac{1}{x}+2)}{\lim_{x\to\infty}(\frac{2}{x}+3)} = \frac{2}{3}.$$

You can also check that

$$\lim_{x\to-\infty} \frac{1+2x}{2+3x} = \frac{2}{3}.$$

Therefore, f only has a single horizontal asymptote and the equation of the line that describes this is $y = \frac{2}{3}$.