Limits at Infinity

This material can be made very rigorous and precise. However, one often gets lost in such abstraction and so we shall view the following material by examples and then present the rigor at the end. The rigor is only presented for those students who wish to understand more about the *why* than just the *how*.

This section will cover the topic of *Limits at infinity*. We shall see that there is a notion of taking x to $\pm\infty$ of some function $f(x)$, and we wish to explore the limiting behavior of this. Let us initiate with two facts and describe what they mean in "English."

1. $\lim_{x\to\pm\infty} \frac{1}{x} = 0$
2. $\lim_{x\to\pm\infty} x = \pm\infty$.

In fact 1, we are saying that as you take x to $+\infty$, then $1/x$ goes to 0. That is, the greater you make x, the smaller you make $1/x$ and in fact, in the limit, we approach 0. Note that taking x to $-\infty$ produces the same effect here; however, this need to be the case in general for arbitrary functions. For example, in fact 2, we find that as we take x greater and greater, x will be sent off to $+\infty$. Similarly, as we take x to be more and more negative, x will be sent off to $-\infty$.

In short, the way one should view taking the limit as x goes to $+\infty$ or $-\infty$ of $f(x)$ is that we to see what happens to $f(x)$ as we make x as great as possible or as negative as possible, respectively.

Limits of polynomials

Let us consider some examples with this in mind.

Example. Find the following limits at infinity.

1. $\lim_{x\to\infty} x^2 + x^3$
2. $\lim_{x\to-\infty} x^2 + x^3$
3. $\lim_{x\to\infty} x - x^2$

Solution.

1. We do the following trick. Note that

$$x^2 + x^3 = x^3(\frac{1}{x^2}) + x^3(1) = x^3\left(\frac{1}{x^2} + 1\right)$$

   and so

$$\lim_{x\to\infty}(x^2 + x^3) = \lim_{x\to\infty} x^3\left(\frac{1}{x^2} + 1\right) = (\lim_{x\to\infty} x^3)(\lim_{x\to\infty}\left(\frac{1}{x^2} + 1\right) = \infty.$$

2. Here we are taking $x \to -\infty$ and so we must view x as becoming arbitrarily negative. We do the following trick. Note that $x^2 + x^3 = x^3(\frac{x^2}{x^3} + \frac{x^3}{x^3}) = x^3(\frac{1}{x} + 1)$ (this will be a common trick in evaluating limits). Therefore, as we have see, since $\frac{1}{x} \to 0$ as $-\infty$, we may conclude

$$\lim_{x\to-\infty} x^2 + x^3 = \lim_{x\to-\infty} x^3(\frac{1}{x} + 1) = \lim_{x\to-\infty} x^3 \lim_{x\to-\infty}(\frac{1}{x} + 1) = -\infty.$$

   If this seems like cheating, it kind of is, but it is mathematically sound. Why this is justified is because for arbitrarily negative x, $x^3(\frac{1}{x} + 1)$ *behaves* like $x^3(0 + 1) = x^3$ because the $\frac{1}{x}$ terms becomes negligible.

3. For this problem, we do a similar trick and find $x - x^2 = x^2(\frac{1}{x} - 1)$. Therefore, taking x to be arbitrarily large, $x - x^2$ behaves like $x^2(0 - 1) = -x^2$ since $\frac{1}{x}$ becomes arbitrarily small. That is

$$\lim_{x\to\infty} x - x^2 = \lim_{x\to\infty} x^2(\frac{1}{x} - 1) = \lim_{x\to\infty}(x^2) \lim_{x\to\infty}(\frac{1}{x} - 1) = \infty(-1) = -\infty.$$

In the previous examples, we saw a trick for finding the limit at infinity of a polynomial. This trick will be used frequently and is stated now.

- If you can, factor out the largest degree term from the polynomial and analyze this new expression's behavior in the limit.

Example. Compute the following limits at infinity.

1. $\lim_{x\to\infty} x^3 + x^2 + x$
2. $\lim_{x\to\infty} -x^4 + x + 2$

Solution. We use the trick from above.

1. $x^3 + x^2 + x = x^3(1 + \frac{1}{x} + \frac{1}{x^2})$ which behaves like x^3 for large x, and so

$$\lim_{x\to\infty} x^3 + x^2 + x = \lim_{x\to\infty} x^3(1 + \frac{1}{x} + \frac{1}{x^2}) = \lim_{x\to\infty} x^3 \lim_{x\to\infty}(1 + \frac{1}{x} + \frac{1}{x^2}).$$

2. $-x^4 + x + 2 = x^4(-1 + \frac{1}{x^3} + \frac{2}{x^4})$, which behaves like $-x^4$ for large x, and so

$$\lim_{x\to\infty} -x^4 + x + 2 = \lim_{x\to\infty} x^4(-1 + \frac{1}{x^3} + \frac{2}{x^4}) \lim_{x\to\infty} x^4 \lim_{x\to\infty}(-1 + \frac{1}{x^3} + \frac{2}{x^4}) = -\infty.$$

Limits of rational functions

We now understand how to limits at infinity for polynomials–just factor out the greatest degree term and analyze. What about rational functions? It turns out there is a similar trick we shall show by example.

Example. Compute the following limits at infinity.

1. $\lim_{x\to\infty} \frac{x+1}{x^2}$
2. $\lim_{x\to\infty} \frac{x^2+1}{2x+1}$
3. $\lim_{x\to\infty} \frac{x^4+x}{2x^4+3}$

Solution. We use a similar trick to before whereby we factor some term.

1. We find that

$$\frac{x+1}{x^2} = \frac{x^2}{x^2}\left(\frac{\frac{1}{x} + \frac{1}{x^2}}{1}\right) = \frac{1}{x} + \frac{1}{x^2}.$$

We note that we could have simply divided through by x^2, but we have done it this way to demonstrate the "trick." We find, then, that

$$\lim_{x\to\infty} \frac{x+1}{x^2} = \lim_{x\to\infty} \frac{1}{x} + \frac{1}{x^2} = 0,$$

since both $\frac{1}{x}$ and $\frac{1}{x^2}$ approach 0 as $x \to \infty$.

2. We do something similar finding

$$\frac{x^2+1}{2x+1} = \frac{x}{x}\left(\frac{x + \frac{1}{x}}{2 + \frac{1}{x}}\right) = \frac{x + \frac{1}{x}}{2 + \frac{1}{x}}.$$

We thus find that as $x \to \infty$, then denominator $2 + \frac{1}{x} \to 2$ and the numerator $x + \frac{1}{x} \to \infty$ and so

$$\lim_{x\to\infty} \frac{x^2+1}{2x+1} = \lim_{x\to\infty} \frac{x + \frac{1}{x}}{2 + \frac{1}{x}} = \frac{\lim_{x\to\infty} x + \frac{1}{x}}{\lim_{x\to\infty} 2 + \frac{1}{x}} = \infty.$$

3. We find
$$\frac{x^4+x}{2x^4+3} = \frac{x^4}{x^4}\left(\frac{1+\frac{1}{x^3}}{2+\frac{3}{x^4}}\right) = \frac{1+\frac{1}{x^3}}{2+\frac{3}{x^4}}.$$
We thus find that as $x \to \infty$, then the denominator $2+\frac{3}{x^4} \to 2$ and the numerator $1+\frac{1}{x^3} \to 1$ and therefore
$$\lim_{x\to\infty} \frac{x^4+x}{2x^4+3} = \lim_{x\to\infty} \frac{1+\frac{1}{x^3}}{2+\frac{3}{x^4}} = \frac{\lim_{x\to\infty} 1+\frac{1}{x^3}}{\lim_{x\to\infty} 2+\frac{3}{x^4}} = \frac{1}{2}.$$

These three examples suggest the following "trick" for taking limits at infinity of rational functions.

Trick for Finding the limit at infinity

If you are given a rational function $p(x)$ (e.g., $\frac{x^2+1}{x^3+1}$), factor out from the numerator and denominator the highest degree term given in the denominator. Fro example, if $p(x) = \frac{x^2+1}{x^3+1}$, the highest degree term in the denominator is x^3. Therefore, we factor from top and bottom
$$\frac{x^2+1}{x^3+1} = \frac{x^3\left(\frac{1}{x}+\frac{1}{x^3}\right)}{x^3(1+\frac{1}{x^3})} = \frac{\left(\frac{1}{x}+\frac{1}{x^3}\right)}{(1+\frac{1}{x^3})}$$
At this point we may take the limit at infinity.

Limits including the exponential

We still need to consider what happens with our functions have exponential terms in them. It turns out that the exponential e^x behaves like the largest degree polynomial, if one existed. We thus have the following results

1. $\lim_{x\to\infty} e^x = \infty$
2. $\lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$
3. $\lim_{x\to\infty} \frac{x^p}{e^x} = 0$ for any p.
4. $\lim_{x\to\infty} \frac{e^x}{x^p} = \infty$ for any p.

We use this to compute limits at infinity with exponentials.

Example. Compute the following limits at infinity.

1. $\lim_{x\to\infty} \frac{1+e^{-x}}{x+2e^{-x}}$
2. $\lim_{x\to\infty} \frac{x^2+3x}{e^x}$
3. $\lim_{x\to\infty} \frac{x+e^{-x}}{2x}$.

Solution. 1. We find
$$\lim_{x\to\infty} \frac{1+e^{-x}}{x+2e^{-x}} = \frac{\lim_{x\to\infty}(1+e^{-x})}{\lim_{x\to\infty}(x+2e^{-x})} = \frac{1+\lim_{x\to\infty} e^{-x}}{\lim_{x\to\infty} x+\lim_{x\to\infty} 2e^{-x}} = 0,$$
since $\lim_{x\to\infty} e^{-x} = 0$.

2. By the fact taht e^x grows faster than any polynomial, taking the limit
$$\lim_{x\to\infty} \frac{x^2+3x}{e^x}$$
is like taking
$$\lim_{x\to\infty} \frac{x^2+3x}{x^p}$$

for some degree $p > 2$. In any case

$$\lim_{x\to\infty} \frac{x^2+3x}{e^x} = \lim_{x\to\infty} \left(\frac{x^2}{e^x} + \frac{3x}{e^x} \right) = \lim_{x\to\infty} \frac{x^2}{e^x} + \lim_{x\to\infty} \frac{3x}{e^x} = 0 + 0 = 0.$$

3. We find

$$\lim_{x\to\infty} \frac{x+e^{-x}}{2x} = \lim_{x\to\infty} \left(\frac{x}{2x} + \frac{e^{-x}}{2x} \right) = \lim_{x\to\infty} \frac{x}{2x} + \lim_{x\to\infty} \frac{1}{2xe^{-x}} = \lim_{x\to\infty} 2 + 0 = 2,$$

where we have used $\lim_{x\to\infty} \frac{1}{2xe^{-x}} = 0$.