Absolute Extrema

Finding absolute extrema on an interval

In this section we concern ourselves with that of absolute extrema. Let f be some function and recall that $f(x)$ is said to be a relative extrema for some value x if, when focused locally enough on x, f appears to have a maximum or minimum in this zoomed in version of the function. We extend this idea to what we usual consider the *maximum* or *minimum* of something.

Definition (Absolute Extrema)**.** Let f be some function defined on some set I and c some point.

1. If $f(x) \leq f(c)$ for all x in I, then $f(c)$ is said to be an **absolute maximum**.

2. If $f(x) \geq f(c)$ for all x in I, then $f(c)$ is said to be an **absolute minimum**.

If $f(c)$ satisfies either condition, then $f(c)$ is said to be an absolute extremum of f on I.

Example. Let $f(x) = x^2$. We concern ourselves in finding the absolute extrema of f on $[-1, 1]$.

Note that $f(x) > 0$ for all points x in $[-1, 1]$ so long as $x \neq 0$, since squaring any nonzero number in $[-1, 1]$ returns a positive number. Moreover $f(0) = 0$ and so, $f(x) \geq f(0)$ for all x in $[-1, 1]$ and so we conclude that $f(0)$ is an absolute minimum of f on $[-1, 1]$.

Next note that $f(x) \leq 1$ for all $x \in [-1, 1]$ since squaring a non-whole number returns a positive non-whole number and $f(\pm 1) = (\pm 1)^2 = 1$. That is to say $f(x) \leq f(\pm 1)$ for all $x \in [-1, 1]$ and so both $f(-1)$ and $f(1)$ are absolute maxima of f on $[-1, 1]$.

It turns out that given a continuous function f and closed interval $[a, b]$ that we may always find absolute extrema.

Theorem (Extrem Value Theorem)**.** *Let f be a continuous function defined on $[a, b]$. Then f attains an absolute extrema on $[a, b]$.*

The only use for this theorem is to check our sanity. For example, if we are given a continuous function f on some closed interval and find no absolute extrema, we know we have done something wrong.

There happens to be a procedure for finding the absolute extrema of continuous functions on closed intervals. We list this out here.

Procedure for finding absolute extrema

Let f be a continuous function on $[a, b]$ and suppose we wish to find the absolute extrema here.

1. Find critical values of f on $[a, b]$

2. Tabulate the relative extremal values of f using this information.

3. Include $f(a)$ and $f(b)$ in this table.

4. Locate the smallest number, which will be the absolute minimum of f on $[a, b]$

5. Locate the largest number, which will be the absolute maximum of f on $[a, b]$.

Example. Find the absolute extrema of the following functions on the indicated intervals.

1. $f(x) = (x + 1)e^x$ on $[-3, 3]$.

2. $g(x) = x + \frac{1}{x}$ on $[1, 2]$

3. $h(x) = x^2 + \ln(x)$ on $[2, 3]$

Solution. We shall just use the procedure enumerated above.

1. First note that f is continuous on $[-3, 3]$ and so must have absolute extrema here. We find $f'(x) = (x+2)e^x)$ by the product rule. Thus $f'(x) = 0$ only when $x = -2$ since e^x is never zero. Thus our only critical value if $x = -2$. We find

$$\begin{aligned} f(-3) &= -2e^{-3} \\ f(-2) &= -e^{-2} \\ f(3) &= 4e^3. \end{aligned}$$

   We find immediately that $f(3)$ is the absolute maximum of f on $[-3, 3]$ since it is the only positive number of the bunch. Next, note that $-e < -2$ and so $-e^{-2} < -2e^{-3}$ by multiplying through by e^{-3} (you could of course use a calculator instead). Therefore, $f(-2) < f(-3) < f(3)$ and so $f(-2)$ is the absolute minimum of f on $[-3, 3]$.

2. Note that g is continuous everywhere on $[1, 2]$–that is, we needn't worry about $x = 0$ since 0 is not in $[1, 2]$. We find $g'(x) = 1 - \frac{1}{x^2}$. Setting $g'(x)$ to zero we find

$$1 - \frac{1}{x^2} = 0$$

   and so, after multiplying through by x^2,

$$x^2 - 1 = 0.$$

   It follows that $g'(x) = 0$ only at $x = 1$ and $x = -1$, but -1 is not in $[1, 2]$, and so we ignore it. We find

$$\begin{aligned} g(1) &= 2 \\ g(2) &= 2 + \frac{1}{2}. \end{aligned}$$

   It follows that $g(1)$ is the absolute minimum of g on $[1, 2]$ and $g(2)$ is the absolute maximum of g on $[1, 2]$.

3. Note that h is continuous on $[2, 3]$ since $\ln(x)$ is continuous on $(0, \infty)$. We find $h'(x) = 2x + \frac{1}{x}$. Setting this to zero

$$2x + \frac{1}{x} = 0$$

   and then multiplying through by x, we find

$$2x^2 + 1 = 0.$$

   Now, since this quadratic does not have any real roots (that is, this equation is never satisfied for any x in $[2, 3]$), h does not have any critical points. Therefore, we need only check the end points of $[2, 3]$. We find

$$\begin{aligned} h(2) &= 2 + \ln(2) \\ h(3) &= 3 + \ln(3). \end{aligned}$$

   We note that $\ln(x)$ is an increasing function and so $h(2) < h(3)$. It follows that $h(2)$ is the absolute minimum of h on $[2, 3]$ and $h(3)$ is the absolute maximum of h on $[2, 3]$.

We next consider finding absolute extrema of some function on an open interval or an unbounded interval. We do so by examples.

Example. Find the absolute extrema of the following functions if they exist.

1. $f(x) = x^2$ on $[0, \infty)$.

2. $f(x) = x^2$ on $(0, \infty)$.

3. $f(x) = x^2$ on $(0, 5]$.

4. $g(x) = x^3 + x^2$ on $(-\infty, \infty)$.

Solution. For the first four examples, we note $f'(x) = 2x$ and the only critical point is $x = 0$.

1. We find that $f(0) = 0$ and $f(x) > 0$ for all $x > 0$. This shows that $f(0)$ is the absolute minimum of f on $[0, \infty)$.However, f does not have any absolute maximum because as $x \to \infty$, $f(x)$ continues to become larger. That is, there is not c in $[0, \infty)$ such that $f(c) \geq f(x)$ for all x in $[0, \infty)$.

2. Here $f(0)$ is never attained since 0 is not in $(0, \infty)$. Furthermore, as $x \to 0$, $f(x)$ becomes arbitrarily small. That is, $\lim_{x \to 0} f(x) = 0$ and so there exists no c such that $f(c) \leq f(x)$ for all $x \in (0, \infty)$. Therefore, with what was stated above, f has no absolute extrema on $(0, \infty)$.

3. Here, we note that f does not have an absolute minimum from the reasoning given above.However, $f(x) \leq f(5)$ for all x in $(0, 5]$ and therefore $f(5)$ is the only absolute extrema of f on $(0, 5]$ and is an absolute maximum.

4. Firstly, $g'(x) = 3x^2 + 2x = x(3x + 2)$ and so $x = 0$ and $x = -\frac{2}{3}$ are the critical points of g. We find

$$g(0) = 0g(-\frac{2}{3}) = \frac{4}{27}.$$

   However,

$$\lim_{x \to \infty} g(x) = \infty \tag{1}$$

$$\lim_{x \to -\infty} g(x) = -\infty, \tag{2}$$

   and so, from (1), we note that g can become arbitrarily large and therefore $g(-\frac{2}{3})$ is not a global extremum. Similarly, from (2), we note that g can become arbitrarily small and therefore $g(0)$ is not a global extremum.

These examples bring up an important point. Firstly, when dealing with an interval not containing one or all its endpoints, the problem becomes a bit more difficult. In these situations we must analyze the problem a bit differently. In either case, we needed to analyze the limiting behavior as x approaches the end point not included in the interval.

Finding absolute extrema with given conditions

In this section, we are interested in analyzing the extremal behavior of some function given some condition. We do this by example.

Example. Solve the extremal problems for the given functions and conditions.

1. Suppose $x + y = 2$. Find x and y such that the expression xe^y is maximized.

2. Suppose $x + y = 2$. Find x and y such that their product is maximized.

Solution. We translate these problems into ones we are familiar with.

1. By $x + y = 2$, we find $y = 2 - x$. Thus, we can make the expression xe^y into a function of x by substituting $2 - x$ for y. So, let $f(x) = xe^{2-x}$. We find

$$f'(x) = e^{2-x} - xe^{2-x} = (1 - x)e^{2-x},$$

   and therefore, since e^{2-x} is never zero, the only critical point of f is $x = 1$. We shall use the second derivative test to determine if $x = 1$ maximize xe^y. Firstly

$$f''(x) = -e^{2-x} - (1 - x)e^{2-x} = (x - 2)e^{2-x}.$$

We find

$$f''(1) = -e,$$

and so $f(1)$ is a relative maximum of f. Lastly, note that

$$\lim_{x\to\infty} f(x) = 0$$
$$\lim_{x\to-\infty} f(x) = -\infty$$

which tells us that $f(1)$ is in fact the absolute maximum of f.

Therefore, since $y = 1$ is the corresponding y value for $x + y = 2$ given $x = 1$, we conclude that $x = 1$ and $y = 1$ are the x and y such that the expression xe^y is maximized given the condition $x + y = 2$.

2. We are given the task of maximizing the product of x and y, namely xy, given the condition $x + y = 2$. We do as we did above by considering xy as a function of x after substituting $2 - x$ for y. Thus, let $f(x) = x(2 - x)$, which is now the function we wish to maximize. Firstly,

$$f'(x) = 2 - 2x,$$

and so $x = 1$ is the only critical point of f. We use the second derivative to determine if $x = 1$ is a relative maximum. We find

$$f''(x) = -2$$

and so, $f''(x) < 0$ showing us that $f(1)$ is in fact a relative maximum. Moreover,

$$\lim_{x\to\infty} f(x) = -\infty$$
$$\lim_{x\to-\infty} f(x) = -\infty,$$

which shows us that $f(1)$ is in fact a global maximum.

Therefore, since $y = 1$ is the corresponding y value for $x + y = 2$ given $x = 1$, we conclude that $x = 1$ and $y = 1$ are the x and y such that their product, xy, is maximized given the condition $x + y = 2$.