Optimization and Modeling

The goal of this material is to be able to, when given some "real world problem", maximize or minimize some quantity. The real difficulty comes from the *modeling* aspect of the problem, not necessarily the *optimizing* aspect. What you will encounter is a list of conditions that you must extract some mathematical relations from. Then you must use these relations to optimize something; viz., you are to maximize or minimize some quantity.

Now, it is common to become frustrated with these problems because their solutions can be rather *ad hoc*. It will become easier with practice so keep trying!

Thus, this section will be taught by example.

Example. Suppose you want to optimize some rectangular shape whose area must be 100 square feet. Further suppose two opposite sides of the rectangle cost \$5 per foot and the other two sides cost \$10 per foot. Your task is to minimize the cost of building such a rectangle.

Solution. The first step to this problem is to build some equations that *model* the problem correctly. Let us name some things. We first note it takes only naming two sides that meet at a corner to specify the rectangle completely. So, let x be the length of one side and y the another side that meets x at a corner. It follows that, since the area of the rectangle must be 100, we must have $xy = 100$; i.e., the product of the sides of the rectangle, xy, must equal the area, 100.

Next, note that opposite sides of a rectangle have equal length and so we may assume that x is the side that costs \$5 per foot and y is the side that costs \$10 per foot. It follows that the price is given by the expression $5x + 5x + 10y + 10y$, where each term appears twice since there is two of each side. That is, price $= 10x + 20y$.

Now, like we did in the previous section, we can rewrite the expression for price as a function of x by solving for y in the condition $xy = 100$ and substituting this for y in the expression for price. Thus,

$$y = \frac{100}{x}$$

and so, if $p(x)$ is to mean price,

$$p(x) = 10x + 20\left(\frac{100}{x}\right) = 10x + \frac{2000}{x}.$$

We have transformed the given problem into a problem of finding the absolute minimum of p. Therefore, we need p' and p'':

$$p'(x) = 10 - \frac{2000}{x^2}$$
$$p''(x) = \frac{4000}{x^3}.$$

We find the critical points of p:

$$10 - \frac{2000}{x^2} = 0$$

gives

$$10x^2 - 2000 = 0$$

and so

$$x^2 = 200$$

which we may conclude $x = \pm\sqrt{2}00$ are the only critical points of p. Moreover, we are dealing with strictly positive values for x since x is a side length and so $x = \sqrt{2}00$ is the only critical value of interest.

We find next that

$$p''(\sqrt{2}00) = \sqrt{2} > 0,$$

from which we conclude $p(\sqrt{2}00)$ is a relative minimum by the second derivative test. Next note that

$$\lim_{x\to\infty} p(x) = \infty$$

and so $p(\sqrt{200})$ is in fact the absolute minimum of p.

Now, given $x = \sqrt{200}$, we conclude the corresponding value for y is $y = 5\sqrt{2}$ by solving for y in $y = \frac{100}{x}$ with $x = \sqrt{200}$. Therefore, a rectangle with side lengths $x = \sqrt{200}$ and $y = 5\sqrt{2}$ are what minimize the price p and are such that $xy = 100$.

Example. Suppose we are to build a rectangular fenced pen off two walls that meet at a right angle. That is, we need only add two walls of fence to create the pen. Suppose further that we are only given 500 feet of fence. What are the dimensions of the two sides needed to maximize the area of the pen?

Solution. We shall again transform the problem into something we can handle by using appropriate modeling. Firstly, we need only name two side of this rectangular fence since there are only two sides to deal with. Let us say x represents the side length of one of the side and y represents the side length of the other. Then, right away, we have xy is the area of the pen since this pen is rectangular; viz., area $= xy$. Next, since we are given 500 feet of fence, we must have that the length used for x and the length used for y must add up to 500; i.e., $x + y = 500$. As we have done before, we may now write the area as a function of x alone. So let $a(x)$ to mean the area of the pen, then,

$$x + y = 500$$

gives us

$$y = 500 - x$$

and so, using the expression for area and substituting for y,

$$a(x) = x(500 - x).$$

Thus the problem has now transformed into finding the global maximum of the function $a(x)$ with the condition $x > 0$ and $a(x) > 0$ since we are dealing with length and area. Now by $a(x) > 0$, we can conclude x must be in $(0, 500)$. This can be seen by the constant sign theorem since $a(x) = 0$ only at $x = 0$ and $x = 500$, and so the intervals of interest are $(0, 500)$ and $(500, \infty)$, from which we find $a > 0$ only on $(0, 500)$. Now, we must find the derivatives of $a(x)$:

$$a'(x) = 500 - 2x \tag{1}$$

$$a''(x) = -2x. \tag{2}$$

By (1), $x = 250$ is a critical point, and by (2), $a(250)$ is a relative maximum since $a''(250) = -2 < 0$, using the second derivative test. Lastly,

$$\lim_{x \to 500} a(x) = a(500) = 0$$

$$\lim_{x \to 0} a(x) = a(0) = 0$$

and so $a(250)$ is in fact the absolute maximum of a. Therefore, we have maximized the area function and so, since $y = 250$ is the corresponding y value give $x = 250$ and $x + y = 500$, we find that the dimensions for maximizing the area of the rectangular pen under the condition $x + y = 500$ are $x = 250$ feet and $y = 250$ feet.