Antiderivatives

In this note we describe what the **antiderivative** is and a couple of its properties. Recall that given a function f, we were usually able to find its derivative f'. Here, we will reverse the process. We will be give some derivative F' known to be identically equal to some function f (i.e., $F'(x) = f(x)$ for all x in the domain of F'), and be tasked with finding what the function F *is*. Put another way, given the function f, we wish to find what function F is such that its derivative F' is identically equal to f. We call such an F an antiderivative (we use "an" instead of "the" for reasons that will become clear later) of f because it is the "undoing" or "opposite" of differentiation. We state all of this as a definition.

Definition (The Antiderivative)**.** If $F'(x) = f(x)$ for all x such that $F'(x)$ exists, then F is called an antiderivative of f.

Example. Let $f(x) = 2x$ and $F(x) = x^2$. Note that $F'(x) = 2x = f(x)$, and so F is an antiderivative of f.

Moreover, suppose $F(x) = x^2 + 5$. Then, again $F'(x) = 2x$, and so F is also an antiderivative of f.

This example brings up an important point: antiderivatives are **not** unique.

Theorem. *Suppose F is an antiderivative of f. Then so is $F + c$ for any real number c.*

Proof. Note that $F'(x) = \frac{d}{dx}(F(x) + c) = \frac{d}{dx}F(x) + \frac{d}{dx}c = F'(x)$, since the derivative of any constant is zero. This shows that $\frac{d}{dx}F(x) = \frac{d}{dx}(F(x) + c) = f(x)$. □

For convenience, if F is such that $F'(x) = f(x)$, then we say $F(x) + c$ is the **general form** of the antiderivative of f. We also introduce some notation. If $F'(x) = f(x)$, then we write $F(x)+c = \int f(x)dx$ and say $\int f(x)dx$ is the **indefinite integral of** f and called $f(x)$ the **integrand**. Let's state this as a definition.

Definition 1 (Indefinite Integral)**.** Suppose $F'(x) = f(x)$. Then we say

$$\int f(x)dx$$

is the indefinite integral of f which is equal to

$$\int f(x)dx = F(x) + c,$$

where c is a constant called the **constant of integration**.

All this really is is just a bunch of notation and terminology for convenience. For example we find $\int F'(x)dx = F(x) + c$, given this notation. This suggests that the **integral sign** $\int$ "reverses" differentiation.

Let's now consider some properties of integration and see corresponding examples.

Properties of Integration

Theorem 1.

1. *$\int x^n dx = \frac{1}{n+1}x^{n+1} + c$, provided $n \neq 1$.*
2. *$\int x^{-1} dx = \ln|x| + c$.*
3. *$\int kf(x)dx = k\int f(x)dx$ for any constant k.*
4. *$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$.*
5. *$\int e^x dx = e^x + c$.*

Properties 1. and 3. are often called the power rule and sum/difference rule, respectively. For property 2., recall that $\frac{d}{dx}\ln|x| = \frac{1}{x}$ and so $\int \frac{1}{x}dx = \ln|x|$. **It is crucial to include the absolute value sign here.** Next, for property 5, we recall that e^x is its own derivative. Property 3. is just as obvious.

Let's go over some examples.

Example. Compute the following indefinite integrals.

1. $\int 5x^2 + x + 1dx$
2. $\int e^x + x^{-0.3}dx$
3. $\int \sqrt{x} + x^{-1}dx$
4. $\int \frac{y+1}{y}dy$

Solution. In order to make these computations, we really need only apply the appropriate properties listed above. We include superfluous steps to aid in fully grasping what's going on.

1.

$$\begin{aligned}\int 5x^2 + x + 1dx &= \int 5x^2dx + \int xdx + \int 1dx \\ &= 5\int x^2dx + \int x^1dx + \int 1x^0dx \\ &= 5 \times \frac{1}{1+2}x^{2+1} + \frac{1}{1+1}x^{1+1} + \frac{1}{1+0}x^{0+1} + c \\ &= \frac{5}{3}x^3 + \frac{1}{2}x^2 + x + c.\end{aligned}$$

2.

$$\begin{aligned}\int e^x + x^{-0.3}dx &= \int e^xdx + \int x^{-0.3}dx \\ &= e^x + \frac{1}{1-0.3}x^{-0.3+1} + c \\ &= e^x + \frac{1}{0.7}x^{0.7} + c \\ &= e^x + \frac{10}{7}x^{7/10} + c\end{aligned}$$

3.

$$\begin{aligned}\int \sqrt{x} + x^{-1}dx &= \int \sqrt{x}dx + \int x^{-1}dx \\ &= \frac{1}{1+\frac{1}{2}}x^{\frac{1}{2}+1} + \ln|x| + c \\ &= \frac{2}{3}x^{\frac{3}{2}} + \ln|x| + c\end{aligned}$$

4.

$$\begin{aligned}\int \frac{y+1}{y}dy &= \int \frac{y}{y}dy + \int \frac{1}{y}dy \\ &= \int 1dy + \int \frac{1}{y}dy \\ &= y + \ln|y| + c\end{aligned}$$