Substitution

In this section we study a technique used in taking integrals that is similar to chain rule. For example, consider the indefinite integral

$$\int 2x\sqrt{x^2+1}dx.$$

How might one proceed in find this antiderivative? This is where **substitution** comes in (colloquially referred to as "u-substitution").

For sake of example, let's pretend all of the following makes perfect sense. Let $u = x^2 + 1$. Then,

$$\frac{du}{dx} = 2x,$$

and, with a bit of pretending, we find

$$du = 2xdx$$

by "multiplying" through by dx. We will not concern ourselves much with why this is legitimate–if it makes you feel better, just treat it as an "calculus trick" for substitution. In any case, from

$$u = x^2 + 1$$

and

$$du = 2xdx$$

we may *substitute* for $2xdx$ and $x^2 + 1$ in

$$\int 2x\sqrt{x^2+1}dx$$

to obtain

$$\int \sqrt{u}du.$$

But we can evaluate this integral:

$$\int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + c.$$

We may now substitute $x^2 + 1$ back for u to obtain

$$\int 2x\sqrt{x^2+1}dx = \int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + c.$$

Let's check to make sure this is actually an antiderivative of $2x\sqrt{x^2+1}$:

$$\frac{d}{dx}\frac{2}{3}(x^2+1)^{\frac{3}{2}} = \frac{2}{3}\frac{3}{2}2x(x^2+1)^{1/2} = 2x\sqrt{x^2+1},$$

as we hoped. Thus we see, through some manipulation and substitution, we can transform a given difficult integral into a rather simple one. This will be the theme for this section: face a difficult integral, decide on a nice substitution, substitute, integrate, then substitute back.

Now that we see the motivation for the technique, we shall explore plenty of examples to get a grasp of what is going on. Getting used to u-substitution takes practice and really relies on one's ability in pattern matching. General rules will be included at the end, but knowing these aren't necessary–one can get by with sufficient practice.

Example. Using substitution, compute the following indefinite integrals

1. $\int (x+1)^5 dx$
2. $\int \frac{x^3}{x^4+1} dx$
3. $\int 2(1+\frac{1}{x})e^{x+\ln|x|} dx$
4. $\int x(x^2+1)^2 e^{(x^2+1)^3} dx$

Solution 1. For these types of problems, we should be asking ourselves, How can we simplify the problem by a clever substitution?

1. We note that we *could* expand $(x+1)^5$ and then integrate term by term. But we are lazy and wish to simplify the problem. So, what *would* be a simplification here? We note that if we set

$$u = x + 1,$$

then

$$du = dx.$$

Substituting this back into the integral, we find

$$\int (x+1)^5 dx = \int (u)^5 du = \frac{1}{6}u^6 + c = \frac{1}{6}(x+1)^6 + c.$$

As a corollary, note that this trick would have worked for *any* power not equal to -1.

We could organize this data into a table

Integrand Term	Substitution
$x+1$	u
dx	du

2. When looking at this problem, seeing the x^4+1 in the denominator should indicate that the final answer should have a natural logarithm. Furthermore, notice that the numerator x^3 is *almost* the derivative of x^4+1. Both of these facts suggest that we should use the following substitution

$$u = x^4 + 1,$$

from which, by differentiating,

$$du = 4x^3 dx.$$

Next, note that in

$$\int \frac{x^3}{x^4+1} dx$$

there isn't a factor of 4 in the numerator that one might hope be there for a direct substitution. However, all is okay because we can simply solve for $x^3 dx$

$$x^3 dx = \frac{du}{4},$$

by dividing through by 4. We may now substitute and integrate

$$\int \frac{x^3}{x^4+1} dx = \int \frac{1}{u}\frac{du}{4} = \frac{1}{4}\ln|u| + c = \frac{1}{4}\ln|x^4+1| + c.$$

Again, we could have organized this into a table

Integrand Term	Substitution
x^4+1	u
x^3dx	$\frac{du}{4}$

3. For this problem, we see that we have an exponential $e^{x+\ln|x|}$ in the integrand. This suggests to us that we should set u to be whatever is power of this exponential. Doing so, we would get

$$u = x + \ln|x|,$$

and, by differentiating,

$$du = (1+\frac{1}{x})dx.$$

This is great because we note that we have a $1+\frac{1}{x}$ term in the integrand as well, which shall allow us to substitute quite nicely. Doing this substitution and then integrating, we would find

$$\int 2(1+\frac{1}{x})e^{x+\ln|x|}dx = \int 2e^u du = 2e^u + c = 2e^{x+\ln|x|} + c.$$

Let's organize some of this into a table

Integrand Term	Substitution
$x+\ln|x|$	u
$(1+\frac{1}{x})dx$	du

4. Similar to the previous problem, we have the exponential $e^{(x^2+1)^3}$ in the integrand, which suggests to set u to be the power of this exponential. This turns out to work. Using this substitution

$$u = (x^2+1)^3$$

we find by differentiating

$$du = 2x(x^2+1)^2dx.$$

where we have used the chain rule. Similar to a previous problem, we see that

$$e^u du = e^{(x^2+1)^3}2x(x^2+1)^2dx$$

doesn't quite match the integrand

$$x(x^2+1)^2e^{(x^2+1)^3}dx$$

because of a factor of 2. To get around this, like before, we solve for

$$x(x^2+1)^2e^{(x^2+1)^3}dx$$

to find

$$\frac{1}{2}e^u du = e^{(x^2+1)^3}x(x^2+1)^2dx.$$

Thus, we substitute and integrate to find

$$\int x(x^2+1)^2e^{(x^2+1)^3}dx = \int \frac{1}{2}e^u du = \frac{1}{2}e^u + c = \frac{1}{2}e^{(x^2+1)^3} + c.$$

General Rules about Substitution

Listed here are some general rules about substitution. You may find these helpful, but understanding substitution this way is not necessary. For some (if not most), learning substitution by practice is enough. In any case, here are the following general rules.

1. When integrating

$$\int f'(x)[f(x)]^p dx$$

for some number $p \neq -1$, set $u = f(x)$. Then $du = f'(x)dx$ and so

$$\int f'(x)[f(x)]^p dx = \int u^p du = \frac{1}{p+1}u^{p+1} + c = \frac{1}{p+1}[f(x)]^{p+1} + c.$$

2. When integrating

$$\int \frac{f'(x)}{f(x)} dx = \int f'(x)[f(x)]^{-1} dx,$$

set $u = f(x)$. Then $du = f'(x)dx$ and so

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{1}{u} du = \ln|u| + c = \ln|f(x)| + c.$$

3. When integrating

$$\int f'(x)e^{f(x)} dx,$$

set $u = f(x)$. Then $du = f'(x)dx$ and so

$$\int f'(x)e^{f(x)} dx = \int e^u du = e^u + c = e^{f(x)} + c.$$

Notice that in all cases, we set $u = f(x)$ and $du = f'(x)dx$. It just so happens there is a most general rule that captures this idea

Theorem 1. *Let f and g be functions such that $g(f(x))$ always makes sense. Then, when integrating*

$$\int f'(x)g(f(x))dx,$$

set $u = f(x)$. Then $du = f'(x)dx$ and so

$$\int f'(x)g(f(x))dx = \int g(u)du.$$

Note that this might not make the integral any easier since we know nothing about g. That is, g might still be a really ugly function and so hopes of integrating may be lost.