The Definite Integral

Introduction

We have seen a rather abstract mathematical object called the indefinite integral in the previous sections. In this section, we shall deal with some a bit more concrete called the **definite integral**. In short, the definite integral is to the indefinite integral what the derivative at a point is to the derivative. That is, both the indefinite integral and the derivative produced a new function, whereas the definite integral and the derivative a point point produced a fixed number.

Recall that if f is a function, the derivative $f'(c)$ at some point c is precisely the slope of the tangent line of the graph of f going through $(c, f(c))$. This was the geometric interpretation of the derivative. Similarly, we have a geometric interpretation of the definite integral. It turns out that the definite integral may be used to compute the (signed) area associated with the graph of some function (colloquially know as "area under the curve"). We will have more to say about this later and shall focus more abstractly on the notion of a definite integral.

Dividing Intervals

Before going on to definite integrals, we need to understand what will be a necessary tool for understanding definite integrals. We must understand how to slice up or divide an interval $[a, b]$ into n slices or subintervals of some equal length. More precisely, given $[a, b]$ and some positive integer n, we wish to divide $[a, b]$ into n subintervals each of length $\Delta x = \frac{b-a}{n}$, where Δx is used as just a name for this length that will be understood as the "change in x" of the interval $[a, b]$ when lying on the x-axis. We show by example what we mean by this.

Example. Consider the interval $[0, 8]$ and suppose we wish to divide this interval into 4 subintervals each of length $\Delta x = \frac{8-0}{4} = 2$. Doing so, we obtain the intervals $[0, 2], [2, 4], [4, 6]$, and $[6, 8]$, where we note each subinterval has length 2 and there are 4 of them.

Now, in anticipation of what is to come, let's use a notational tool. We can label the endpoints by $a = 0 = x_0$, $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, and $b = x_4 = x_n = 8$. That is, given any interval $[a, b]$ divided into n subintervals, we may pick out the $n + 1$ endpoints, two corresponding to each subinterval, and label them by $a = x_0$, $x_1, x_2, \ldots, x_n = b$.

Let us reiterate the statement at the end of the example. Suppose we are given an interval $[a, b]$ and a positive integer n. Suppose further that we have divided $[a, b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$. Then we shall label the $n + 1$ endpoints of these subintervals by $a = x_0, x_1, x_2, \ldots, x_n = b$.

Lastly, let's see how to obtain the subintervals given $[a, b]$ and n.

Example. Consider the interval $[2, 8]$ and suppose we want to divide this interval into 6 subintervals of length $\Delta x = \frac{8-2}{6} = 1$. To construct this division, we sequentially add 1 to the endpoint 2 until we reach 8. That is, our first interval is $[2, 2 + 1] = [2, 3]$. Then, our next is $[3, 3 + 1] = [3, 4]$. Continuing, we obtain the subintervals $[2, 3], [3, 4], [4, 5], [5, 6], [6, 7]$, and $[7, 8]$.

Summation

Before we see the definite integral, we need to define a very useful symbol $\sum$ called the summation sign. Let's see a formal definition first.

Definition (Summation)**.** Let g be some function defined on the positive integers. Then we define the symbol $\sum$ by

$$\sum_{k=1}^{n} g(k) = g(1) + g(2) + \cdots + g(n),$$

i.e., $\sum\limits_{k=0}^{n} g(n)$ is the sum of n terms, starting with $g(1)$, and ending with $g(n)$.

Example. Let g be the function defined by $g(x) = x$ and suppose we wish to compute $\sum_{k=1}^{5} g(k)$. To do so, we simply apply the definition

$$\sum_{k=1}^{5} g(k) = g(1) + g(2) + g(3) + g(4) + g(5) = 1 + 2 + 3 + 4 + 5 = 15.$$

Example. Now let $\Delta x = \frac{1}{4}$ and $g(x) = x^2$, and suppose we wish to compute $\sum_{k=1}^{4} g(k)\Delta x$. Again, we just apply the definition

$$\sum_{k=1}^{4} g(k)\Delta x = g(1)\Delta x + g(2)\Delta x + g(3)\Delta x + g(4)\Delta x = \frac{1}{4} + \frac{4}{4} + \frac{9}{4} + \frac{16}{4} = \frac{15}{2}.$$

We now combine our understanding of dividing intervals and summation into one mathematical object. Suppose we are given an interval $[a, b]$ which is divided into n equal length subintervals of length $\Delta x = \frac{b-a}{n}$. Suppose further we have labeled the endpoints of these subintervals by $a = x_0, x_1, x_2, \ldots, x_n = b$. Lastly, suppose we are give some function f defined on $[a, b]$. We may now defined the **right hand sum** and **left hand sum** of f on $[a, b]$ of order n by

$$\textbf{Right Hand Sum}: \sum_{k=1}^{n} f(x_k)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

$$\textbf{Left Hand Sum}: \sum_{k=0}^{n-1} f(x_k)\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x.$$

Notice that the left hand sum starts at x_0 and ends at x_{n-1} and the right hand sum starts at x_1 and ends at x_n; i.e., the right hand sum starts shifted to the right by one index from the left hand sum. This note should serve as a suggestion to the sums' respective names. Note that these sums are sometimes referred to as **Riemann sums**.

Let's see an example.

Example. Let $f(x) = x^2$ be defined on $[0, 8]$ and divide this interval into 4 subintervals of length $\Delta x = \frac{8-0}{4} = 2$ (note that we have implicitly assumed $n = 4$). Next, label the endpoints of these subintervals as done above. We wish to compute the right and left hand sums of this function on the given divided interval.

For the right hand sum,

$$\begin{aligned}\sum_{k=1}^{4} f(x_k)\Delta x &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= f(2)\Delta x + f(4)\Delta x + f(6)\Delta x + f(8)\Delta x \\ &= \frac{4}{4} + \frac{16}{4} + \frac{36}{4} + \frac{64}{4} = 30.\end{aligned}$$

For the left hand sum,

$$\begin{aligned}\sum_{k=0}^{3} f(x_k)\Delta x &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= f(0)\Delta x + f(2)\Delta x + f(4)\Delta x + f(6)\Delta x \\ &= \frac{0}{4} + \frac{4}{4} + \frac{16}{4} + \frac{36}{4} = 14.\end{aligned}$$

The Definite Integral

We may now define what the definite integral *is*. Suppose we have a function f continuous on $[a, b]$ and let $\Delta x = \frac{b-a}{n}$. Dividing the interval into n subintervals of length Δx and labeling the endpoints as we have done above, we notice that the left and right hand sums

$$\sum_{k=1}^{n} f(x_k)\Delta x, \qquad \sum_{k=0}^{n-1} f(x_k)\Delta x$$

are suitable for taking the limit as $n \to \infty$. It is a matter of fact that, since f is continuous on $[a, b]$, that these limits as $n \to \infty$ exist and satisfy

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x.$$

We call the common value of these limits the definite integral of f from a to b and denote this definite integral by

$$\int_a^b f(x)dx.$$

We shan't deal further with this definition until the next section.

Definite Integral As Area

In this section we consider the definite integral as "area under the curve." As an example, consider the linear function $y = 2x$ on the interval $[0, 4]$. We note that this define a triangle with vertices $(0, 0), (4, 8)$, and $(4, 0)$. If you imagine taking successive left and right hand sums of $y = 2x$ on $[0, 4]$, we see that we approximate the area of this triangle. In the limit, we in fact get precisely this area. It follows that

$$\int_0^4 2xdx = \frac{1}{2} \times 4 \times 8 = 16,$$

since the base of the triangle is of length 4 and the height is of length 8.