The Fundamental Theorem of Calculus

In the last section we defined what a definite integral *is*, but not how to compute it. The computation of definite integrals is the goal of this section. We shall face what is called **The Fundamental Theorem of Calculus**, which allows us to compute the definite integral of a function f provided we know an antiderivative of it.

Properties of the Definite Integral

Before seeing this theorem, let's first collect some properties of the definite integral into a single theorem.

Theorem. *Let f and g be continuous function on some interval $[a, b]$. Then the following hold.*

1. *If $f(x) \geq 0$, then $\int_a^b f(x)dx \geq 0$.*
2. *If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.*
3. *If $m \leq f(x) \leq M$ on $[a, b]$, then*

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

4. *$\int_a^a f(x)dx = 0$.*
5. *$\int_a^b kf(x)dx = k\int_a^b f(x)dx$ for any constant k.*
6. *$\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.*
7. *$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, provided $a < c < b$.*
8. *$\int_b^a f(x)dx = -\int_a^b f(x)$, provided $a < b$.*

Note that we are not quite ready to apply any of these properties at our will. To do so, we need the fundamental theorem of calculus.

The Fundamental Theorem of Calculus

We may now explore the fundamental theorem of calculus. The theorem comes in two flavors, both typically called The Fundamental Theorem of Calculus. We state both in a single theorem.

Theorem (The Fundamnetal Theorem of Calculus)**.** *Suppose f is continuous on $[a, b]$.*

1. *Let x be in $[a, b]$ and let*

$$g(x) = \int_a^x f(t)dt.$$

   Then $g'(x) = f(x)$.

2. *If F is any antiderivative of f, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Notice that the first flavor does not do much for computing definite integrals but is rather a property of definite integrals. However, the second flavor is precisely what we need to compute definite integrals and is what shall be used from now on.

Examples

Example. Compute the following definite integrals.

1. $\int_0^5 x^2 dx$.
2. $\int_1^b \sqrt{2x+3}dx$ for some b such that $2b+3 \geq 1$.
3. $\int_3^4 \frac{1}{x+2} dx$.

Solution. The general procedure for solving these problems is to compute an indefinite integral of integrand in question and then use the Fundamental Theorem of Calculus.

1. We first compute an indefinite integral of x^2. We recall

$$\int x^2 dx = \frac{1}{3}x^3 + c$$

and so $F(x) = \frac{1}{3}x^3$ is an indefinite integral of x^2. It follows by the Fundamental Theorem of Calculus that

$$\int_0^5 x^2 dx = F(5) - F(0) = \frac{1}{3}5^3 - 0 = \frac{5^3}{3}.$$

2. We compute an indefinite integral first. However, notice that we must use substitution first, so let $u = 2x+3$. Then, $du = 2dx$, and hence $dx = \frac{du}{2}$. Therefore,

$$\int \sqrt{2x+3}dx = \int \sqrt{u}\frac{du}{2} = \frac{1}{2}\frac{2}{3}u^{3/2} + c = \frac{1}{3}(2x+3)^{3/2} + c.$$

That is, $F(x) = \frac{1}{3}(2x+3)^{3/2}$ is an antiderivative of $\sqrt{2x+3}$. Thus, by the Fundamental Theorem of Calculus,

$$\int_1^b \sqrt{2x+3}dx = F(b) - F(1) = \frac{1}{3}(2b+3)^{3/2} - \frac{1}{3}5^{3/2}.$$

3. Again, compute an indefinite integral of the integrand $\frac{1}{x+2}$. We must use substitution, so set $u = x+2$. Then $du = dx$. It follows that

$$\int \frac{1}{x+2}dx = \int \frac{1}{u}du = \ln|u| + c = \ln|x+2| + c.$$

Therefore, $F(x) = \ln|x+2|$ is an antiderivative of $\frac{1}{x+2}$. Thus, by the Fundamental Theorem of Calculus,

$$\int_3^4 \frac{1}{x+2}dx = F(4) - F(3) = \ln 6 - \ln 5 = \ln\frac{6}{5}.$$