1.3 Exponential Models

Definitions

Definition 1 (Compound Interest)**.** *Suppose a principal P earns interest at the annual rate of r, and interest is compounded m times a year. Then F after t years is*

$$F = P(1+i)^n = P(1+\frac{r}{m})^{mt}$$

where $n = mt$=number of time periods and $i = r/m$ =interest per period.

Definition 2 (Present Value)**.** *Given $F = P(1+r/m)^{mt}$ and say we want to know how many dollars P to set aside so we will have a future amount of F dollars after t years. We solve for P:*

$$P = \frac{F}{(1+r/m)^{mt}} = \textit{Present value}$$

Definition 3. *If a principal P earns interest at an annual rate of r, and interest is compounded continuously, then F after t years is $F = Pe^{rt}$.*

Definition 4 (Present Value for Continuous Compounding)**.** *Present value for continuous compounding is given by $P = Fe^{-rt}$ = Present value.*

1.5 Logarithms

Definitions

Definition 5 (Logarithm)**.** *Let $a > 0$ and $a \neq 1$. If $x > 0$, then the logarithm base a of x, denoted $\log_a x$ is defined as follows:*

$$y = log_a x \text{ if and only if } x = a^y$$

Definition 6 (Common Logarithm/Log Base 10)**.** *Let $a = 10$, then*

$$y = \log_{10} x = \log(x) \text{ if and only if } x = 10^y$$

Definition 7 (Natural Log/Log Base e)**.** *Let $a = e$, then*

$$y = \log_e x = \ln(x) \text{ if and only if } x = e^y$$

Properties of Logarithms

1. $a^{\log_a x} = x$ if $x > 0$ (so exponentiation undoes the logarithm).
2. $\log_a a^x = x$ for all x (so logarithm undoes exponentiation).
3. $\log_a xy = \log_a x + \log_a y$
4. $\log_a x/y = \log_a x - \log_a y$
5. $\log_a x^c = c \log_a x$
6. Let $a > 0$, then $a^x = (a)^x = (e^{\ln a})^x = e^{x \ln a}$
7. Change of Base: $\log_a x = \frac{\log_b x}{\log_b a}$.

Domain of a logarithm

Note that the logarithm function $\log_a(x)$ is defined on when its "input" x is positive. So, if we have $\log_a(x + 1)$, the domain of this function is $(-1, \infty)$ since if $x > -1$, $x + 1$ is positive.

Examples Worked Out

Example 1. Compute $5^{3 \log_5 6}$.

Solution 1. We must first use property 5. to get $5^{3 \log_5 6} = 5^{\log_5 6^3}$. Then by property 1., we have $5^{\log_5 6^3} = 6^3 = 216$, giving us the computation.

Example 2. Solve for x in $e^{-x} = 10$.

Solution 2. Recall $\ln(e^a) = a$ for any a and so, taking the natural log on both sides of the equation, we get $\ln(e^{-x}) = -x = \ln(10)$, and so $x = -\ln(10) = \ln \frac{1}{10}$.

Example 3. Solve for x in $2 \log_2 x = 3$.

Solution 3. Before we can simplify, we must move the 2 out front into the logarithm: $2 \log_2 x = \log_2 x^2$ by property 5. above. Thus we have $\log_2 x^2 = 3$. Now, using property 1., we get $2^{\log_2 x^2} = x^2 = 2^3$, and so $x = \sqrt{2^3} = 2\sqrt{2}$.

Example 4. Solve for x in $6 \cdot 8^x = 10$.

Solution 4. We first divide both sides by 6 to get $8^x = \frac{10}{6}$. Then we use property 2. by taking $\log_8$ on both sides: $\log_8 8^x = x = \log_8 \frac{10}{6} = \log_8 \frac{5}{3}$.

Example 5. Solve for x in $\log_3 x = \log_3 5 + \log_3 7 - 2 \log_3 4$.

Solution 5. To solve this problem, we must first make some simplifications on the right hand side of the equation by using properties 3., 4., and 5.. Let's combine $\log_3 5 + \log_3 7$ first. We have from property 3. that $\log_3 5 + \log_3 7 = \log_3(5 \cdot 7) = \log_3 35$. Thus we have now $\log_3 x = \log_3 35 - 2\log_3 4$. Let's simplify $2\log_3 4$: note we can use property 5. to get $2\log_3 4 = \log_3 4^2 = \log_3 16$. Thus we have now $\log_3 x = \log_3 35 - \log_3 16$. Let's simplify $\log_3 35 - \log_3 16$: note that we can use property 4. to get $\log_3 35 - \log_3 16 = \log_3 \frac{35}{16}$. Thus we have $\log_3 x = \log_3 \frac{35}{16}$. We may now use property 1. to get $3^{\log_3 x} = x = 3^{\log_3 \frac{35}{16}} = \frac{35}{16}$.

Example 6. Solve for x in $\log(x+1) = 1 + \log x$.

Solution 6. Recall here that log means "log base 10". First we move all the things containing x to one side by subtracting by $\log x$ from both sides to get: $\log(x+1) - \log(x) = 1$. Then we use property 4. to get $\log(x+1) - \log(x) = \log \frac{x+1}{x} = 1$. We now use property 1. to get $10^{\log \frac{x+1}{x}} = \frac{x+1}{x} = 10^1 = 10$. From here we solve for x as usual to get $x = 1/9$.

Example 7. Find the domain of $\log(-x+1)$.

Solution 7. Recall that logarithm functions are only defined for when the "inside" is greater than 0. Thus we need $-x+1 > 0$ and so $1 > x$; i.e., the domain is $(-\infty, 1)$.