3.3 The Derivative

Definitions

Definition 1 (The Derivative)**.** If $y = f(x)$, the derivative of $f(x)$, denoted by $f'(x)$ is defined to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists.

Note: you might see y', $\frac{dy}{dx}$, and $\frac{d}{dx}f(x)$ to mean the derivative.

Examples Worked Out

Finding Derivatives

Example 1. Find the derivative of $f(x) = \frac{1}{x}$.

Solution 1. We directly apply the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to find $f'(x)$. However, let us compute $f(x+h)$ first:

$$f(x+h) = \frac{1}{x+h},$$

where we have just evaluated f at the "number $x + h$. Then,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h \to 0} \frac{1}{h(x+h)} - \frac{1}{xh} \\
&= \lim_{h \to 0} \left(\frac{x}{xh(x+h)} - \frac{x+h}{xh(x+h)} \right) \\
&= \lim_{h \to 0} \frac{-h}{xh(x+h)} \\
&= \lim_{h \to 0} \frac{-1}{x(x+h)} \\
&= -\frac{1}{x^2}.
\end{aligned}$$

Example 2. Let $f(x) = \frac{1}{x+a}$ where a is some fixed real number and find $f'(x)$.

Solution 2. Note that we can basically just apply the work above with the change being instead of 'x's everywhere, we have '$x + a$'s everywhere:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+a+h} - \frac{1}{x+a}}{h} \\
&= \lim_{h\to 0} \frac{1}{h(x+a+h)} - \frac{1}{(x+a)h} \\
&= \lim_{h\to 0} \left(\frac{x+a}{(x+a)h(x+a+h)} - \frac{x+a+h}{(x+a)h(x+a+h)} \right) \\
&= \lim_{h\to 0} \frac{-h}{(x+a)h(x+a+h)} \\
&= \lim_{h\to 0} \frac{-1}{(x+a)(x+a+h)} \\
&= -\frac{1}{(x+a)^2}.
\end{aligned}
$$

Finding Tangent Line

Example 3. Let $f(x) = \frac{1}{x}$ and find the tangent line to the graph of $f(x)$ at $(2, f(2))$.

Solution 3. Recall that the tangent line to a graph at some point $(c, f(c))$ is given by the linear equation that goes through the point $(c, f(c))$ and has slope

$$m_{tan}(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h};$$

i.e., the slope of the tangent line is the derivative at c. Thus, if we find $m_{tan}(2)$, we can simply use the point slope form given this slope and the point $(2, f(2))$ to find the equation of the tangent line. We first find $m_{tan}(2)$ by the example above:

$$
\begin{aligned}
m_{tan}(2) &= \lim_{h\to 2} \frac{f(2+h) - f(2)}{h} \\
&= f'(2) \\
&= -\frac{1}{2^2} \\
&= -\frac{1}{4}.
\end{aligned}
$$

This gives us the slope of the equation for the tangent line and so we need only use the point slope form to find the equation for the tangent line. Recall that the point slope form is the equation $y - y_0 = m(x - x_0)$, where this gives the equation for the line through (x_0, y_0) with slope m. For this example, we get

$$m = m_{tan}(2),$$

$$x_0 = 2,$$

$$y_0 = f(2) = \frac{1}{2}.$$

Thus, our answer is given by $y - \frac{1}{2} = -\frac{1}{4}(x-2)$, and solving for y, $y = -\frac{x}{4} + 1$.

To be sure, we check our answer by plugging in $x = 2$ to make sure this line hits the point $(2, f(2)) = (2, 1/2)$. We get $y = -\frac{2}{4} + 1 = -\frac{1}{2} + 1 = \frac{1}{2}$, showing us the line does in fact go through $(2, 1/2)$.