Constant Sign Theorem

Recall the following theorem

Theorem. *If f is continuous and non-zero on (a, b), then f has the same sign on (a, b).*

We show how to use this theorem by the following example.

For continuous functions

Let $f(x) = x^2 - 1$ and find where f is positive or negative. Note that f is continuous everywhere and $f(x) = 0$ only at $x = -1$ and $x = 1$. Therefore, the intervals we must consider for using the constant sign theorem are $(-\infty, -1), (-1, 1), (1, \infty)$. We find that $f(-2) = 4 - 1 > 0$ and so f is positive on $(-\infty, -1)$ by the constant sign theorem. Next, $f(0) = -1 < 0$ and so f is negative on $(-1, 1)$. Lastly, $f(2) = 4 - 1 > 0$ and so f is positive on $(1, \infty)$.

Finding where increasing/decreasing

Find where $f(x) = x^2 - 1$ is increasing or decreasing. Firstly, $f'(x) = 2x$ and so, since $f'(x) = 0$ only at $x = 0$ and f' is continuous everywhere, in order to use the constant sign theorem, we consider only the intervals $(-\infty, 0)$ and $(0, \infty)$. We find $f'(-1) = -2 < 0$ and so f' is negative (and hence f is decreasing) on $(-\infty, 0)$. Next, $f'(1) = 2 > 0$ and so f' is positive (and hence f is increasing) on $(0, \infty)$.

Finding where concave up/concave down

Find where $f(x) = x^3 + x$ is concave up or down. We find $f'(x) = 3x^2 + 1$ and $f''(x) = 6x$. Note f'' is continuous everywhere and $f''(x) = 0$ only $x = 0$. Thus, in order to use the constant sign theorem, we need only consider $(-\infty, 0)$ and $(0, \infty)$. We find $f''(-1) = -6 < 0$ and so f'' is negative (and hence f is concave down) on $(-\infty, 0)$. Next, $f''(1) = 6 > 0$ and so f'' is positive (and hence f is concave up) on $(0, \infty)$.