Constant Sign Theorem

Theory

In this note is included the *Constant Sign Theorem* and examples. The theorem is the following.

Theorem (Constant Sign Theorem)**.** *Suppose f is continuous and nonzero on some interval (a, b). Then f is either always positive on (a, b), or always negative on (a, b); i.e., f has constant sign on (a, b).*

To use it, we find where f is continuous, where f is zero, and then choose the largest intervals from this information such that f is nonzero and continuous on these intervals. Next, we pick a point in each of these intervals, test the whether f is positive or negative at these points, and then apply the previous theorem to conclude the sign of f on each of these intervals. This theorem finds use in

1. Finding where a function is positive/negative
2. Finding where a function is increasing/decreasing
3. Finding where a function is concave up/concave down

provided the function of interest is nice enough (differentiable at most places, etc.). In reality, each of these applications are really the same thing just applied to different functions. This is depicted in the following table.

Function used	Use	How it's used
f	Find where f is positive or negative	Find where $f(x) > 0$ or $f(x) < 0$
f'	Find where f is is increasing or decreasing	Find where $f'(x) > 0$ or $f'(x) < 0$
f''	Find where f is concave up or concave down	Find where $f''(x) > 0$ or $f''(x) < 0$

Application

Throughout, let $f(x) = x^2 - 1$.

Where f is positive or negative

We must first find where f is continuous and nonzero. Note that f is a polynomial and so is continuous everywhere. Second, setting $f(x) = 0$, we find $x^2 - 1 = 0$ and thus $x = \pm 1$. Therefore, since f is continuous everywhere and zero only at $x = -1$ and $x = 1$, the intervals we are interested in are $(-\infty, -1), (-1, 1)$, and $(1, \infty)$. That is, these intervals are the largest for which the function f is nonzero and continuous.

Next, we test points from each of these intervals and then apply the *Constant Sign Theorem*. Let's organize this into a table.

Interval	Test Point	Value of f	Conclusion
$(-\infty, -1)$	-2	$f(-2) = 3$	f is positive on $(-\infty, -1)$
$(-1, 1)$	0	$f(0) = -1$	f is negative on $(-1, 1)$
$(1, \infty)$	2	$f(2) = 3$	f is positive $(1, \infty)$

Let's explain what happened here. Considering the interval $(-\infty, -1)$, we test the point -2 since this point is in the interval. We find $f(-2) = 3$, which is a positive number, and so, since the constant sign theorem tells us f *must* have the same sign on this interval, we conclude f is positive on $(-\infty, -1)$. We apply the same reasoning for the other two intervals.

Where f is increasing or decreasing

Recall that f is increasing when $f'(x) > 0$ and f is decreasing when $f'(x) < 0$. Thus, we may apply the work in the previous example to the new function $f'(x) = 2x$. Let's see it worked out here.

We note that $f'(x) = 0$ only when $x = 0$ since this is the only value for which $2x = 0$. Moreover, f' is a polynomial and so f' is continuous everywhere. It follows that the largest intervals for which the function f' is continuous and nonzero are the intervals $(-\infty, 0)$ and $(0, \infty)$.

Now, let us test points from each of these intervals as we did above and organize the information into a table.

Interval	Test Point	Value of f'	Conclusion
$(-\infty, 0)$	-1	$f'(-1) = -2$	f is decreasing on $(-\infty, 0)$
$(0, \infty)$	1	$f'(1) = 2$	f is increasing on $(0, \infty)$

Let's explain what happened here. Considering the interval $(-\infty, 0)$, we tested the point -1 since this point is in the interval. We find $f'(-1) = -2$, which is a negative number, and so, since the constant sign theorem tells us f' *must* have the same sign on this interval, we conclude f' is negative on $(-\infty, 0)$. We apply the same reasoning for the other interval.

Notice how I practically copied and pasted the work from the previous example. I was able to do this because f' *is* just another function and so the constant sign theorem applies just the same. The only difference is that since we applied the constant sign theorem to f' instead of f, we earned different information about f for our work.

Where f is concave up or concave down

I will provide two examples for determining where a function is concave up or concave down. I do this because finding where $f(x) = x^2 - 1$ is concave up or concave down turns out to be trivial, but is included for completeness.

Recall that f is concave up provided $f''(x) > 0$, and concave up provided $f''(x) < 0$. Thus, we need to find f''. But, from $f'(x) = 2x$, we find immediately that $f''(x) = 2$. That is, f'' is constantly the value 2, which is a positive number. Therefore, since f'' is continuous everywhere and in fact always positive, by the constant sign theorem, $f''(x) > 0$ on $(-\infty, \infty)$. We conclude that f is always concave up.

Where a different function is concave up or concave down

The method here is precisely the same as the first two examples: find where the function is nonzero and continuous and apply the constant sign theorem. Let $f(x) = x^3$ and let's cut to the chase and find f'':

$$f'(x) = 3x^2 \text{ and so } f''(x) = 6x.$$

Therefore, $f''(x) = 0$ only at $x = 0$ since this is the only value for which $6x = 0$. Moreover, f'' is continuous everywhere by being a polynomial. It follows that the largest intervals for which the function f'' is continuous and nonzero are the intervals $(-\infty, 0)$ and $(0, \infty)$.

Let's organize the useful data in a table as before.

Interval	Test Point	Value of f''	Conclusion
$(-\infty, 0)$	-1	$f''(-1) = -6$	f is concave down $(-\infty, 0)$
$(0, \infty)$	1	$f''(1) = 6$	f is concave up $(0, \infty)$

The reasoning for where f is concave up or concave down is as before.