Linear Approximation

In this note is an explanation on what *linear approximation* is and how to use it.

Theory

Recall that the derivative of a function f at a point c is given by some limit

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}.$$

We can rewrite this limit as

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

by the substitution $h = x - c$, with $x \neq c$ ever. Because $f'(c)$ is defined as such a limit, we may approximate $f'(c)$ by taking x close to c. Doing this, we would get

$$f'(c) \approx \frac{f(x) - f(c)}{x - c}.$$

We may now solve for $f(x)$:

$$f(x) \approx f'(c)(x - c) + f(c).$$

That is, given the rate of change of f at some point c (that is, $f'(c)$), the difference of x and c (that is, $x - c$), and f evaluated at c (that is, $f(c)$), we may give an approximation for computing $f(x)$. This approximation is called *linear approximation* because the right hand side defines a linear polynomial; i.e., it defines a line.

Application

Let us see two types of examples where linear approximation can be used.

Approximation a function at a difficult point

Here we focus on approximating the value of a function at some point making use of the function and its derivative at an "easy" point.

Example 1. Approximate the following computations linearly.

1. $\sqrt{24.5}$
2. $\ln 1.2$

Solution 1. The idea will always be the same. First we must choose a value that we know how to evaluate the function with. For example, in the first example, $\sqrt{25}$ is easy to evaluate, so we shall use 25 to approximate $\sqrt{24.5}$. That is, we shall use 25 as our "c" value to approximate our "x" value of 24.5.

1. Here we will let $f(x) = \sqrt{x}$ and then note that $f(25)$ is easily calculated and that we wish to approximate $f(24.5)$. To do so, we use the linear approximation detailed in the *Theory* section. So we must first find the derivative:

   $$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}}.$$

   Note that even $f'(25)$ is easy to calculate. Thus, since we may calculate $f(25)$ and $f'(25)$ just fine, we use $c = 25$ as our approximation point.

   Now, to approximate $f(24.5)$, we need only evaluate the right hand side of

   $$f(24.5) \approx f'(25)(24.5 - 25) + f(25).$$

We find

$$f(25) = \sqrt{25} = 5$$
$$f'(25) = \frac{1}{2}\frac{1}{\sqrt{25}} = \frac{1}{10}$$
$$24.5 - 25 = \frac{1}{2}.$$

Therefore,

$$f(24.5) \approx f'(25)(24.5 - 25) + f(25) = -\frac{1}{10} \times \frac{1}{2} + 5 = 24.5.$$

We can also organize this data into a table if it helps.

x	c	$f(c)$	$f'(c)$	$x - c$	$f(x) \approx f'(c)(x-c) + f(x)$
24.5	25	$f(25) = 5$	$f'(25) = \frac{1}{2} \times \frac{1}{25} = \frac{1}{10}$	$24.5 - 25 = \frac{1}{2}$	$f(24.5) \approx -\frac{1}{10} \times \frac{1}{2} + 5 = 4.95$

So, in short, what we did was determined what the function was that we wanted to approximate, find a value for which the function can easily be evaluated at, computed the derivative and the function at this nice point, and then used linear approximation.

Computing $\sqrt{24.5}$ with a calculator, we find $\sqrt{24.5} \approx 4.94$, and so our linear approximation did a good job.

2. Here we let $f(x) = \ln(x)$ and note that $\ln 1$ is easy to evaluate since $\ln 1 = 0$. So we shall use 1 as our "c" value to approximate our "x" of 1.2. But first, we must find the derivative of f:

$$f'(x) = \frac{1}{x},$$

which is still easy to compute at 1 since $f'(1) = \frac{1}{1} = 1$. Lastly, we compute $x - c = 1.2 - 1 = \frac{1}{5}$. We thus have all we need to approximate $f(1.2)$ linearly:

$$f(1.2) \approx f'(1)(1.2 - 1) + f(1) = 1 \times \frac{1}{5} + 0 = \frac{1}{5} = 0.2.$$

We can again organize this work into a table.

x	c	$f(c)$	$f'(c)$	$x - c$	$f(x) \approx f'(c)(x-c) + f(x)$
1.2	1	$f(1) = 0$	$f'(1) = 1$	$1.2 - 1 = 0.2 = \frac{1}{5}$	$f(1.2) \approx 1 \times \frac{1}{5} + 0 = \frac{1}{5}$

Computing $\ln 1.2$ with a calculator, we find $\ln 1.2 \approx 0.18$, and so our linear approximation did a good job.

Approximating the change in something

What we will see here is similar to above but rather than approximating the "$f(x)$", we shall approximate the "$f'(c)$". It turns out that doing so will approximate the change in something because, as you recall, the derivative measures the rate of change of some function. Therefore, say if we were to approximate the change in area A of a square whose side length goes from c to x, we may use

$$\text{Approximate change in area from } c \text{ to } x = A(x) - A(c) \approx A'(c)(x - c).$$

That is, given the rate of change of area at c, we may approximate the change in area. Let's see this as a precise example.

Example 2. The side length of a square increases from 4 inches to 4.2 inches. Use linear approximation to approximate the change in area.

Solution 2. Here we shall choose the appropriate function for area so that we can use the linear approximation method from above. Next note that since the side length of the square changes from 4 to 4.2, our "c" value is 4 and our "x" value is 4.2. This is because we want to measure the change in area as an increase; i.e., we want to measure the change in area as area goes from $A(4)$ to $A(4.2)$. That is to say, we want to measure $A(4.2) - A(4)$, so, in order to use the linear approximation from above, we must use $c = 4$ and $x = 4.2$. Besides, $A'(c)$ will be much easier to compute. Let A be the area of the square and note that if y is the side length, then certainly $A(y) = y^2$. Moreover, $A'(y) = 2y$. Next, $A'(4) = 2 \times 4 = 8$. Thus, we have all we need to linearly approximate the change in area:

$$A(4.2) - A(4) \approx A'(4)(4.2 - 4) = 8 \times \frac{1}{5} = \frac{8}{5}.$$

Therefore, the change in area is approximately $\frac{8}{5} = 1.6$.

Computing the change in area with a calculator, we find $A(4.2) - A(4) = 1.64$, and so our linear approximation did a good job.