Practice Exam 2

Problem 1. Find

(a) Critical points

(b) Where the function is positive

(c) Where the function is increasing or decreasing

(d) Where the function is concave up or concave down

(e) Locate any points of inflection

(f) Locate any relative extrema

for

1. $f(x) = \frac{x^3}{3} + x^2 + x$
2. $g(x) = \frac{x^2}{x+1}$
3. $e^{2x}(x-1)$

Solution 1.

1. (a) $f'(x) = x^2 + 2x + 1 = (x+1)^2$ and so $f'(x) = 0$ at $x = -1$ only. Therefore $x = -1$ is the only critical point of f.

   (b) In order to do this problem, we can use the constant sign theorem. We note $f(x) = x(\frac{x^2}{3}+x+1)$ and cannot be factored further. Thus $f(x) = 0$ only at $x = 0$. It follows that f is nonzero and continuous only on the intervals $(-\infty, 0)$ and $(0, \infty)$, which are thus the intervals of interest. We find $f(-1) = -\frac{1}{3} < 0$ and $f(1) = \frac{7}{3} > 0$. Therefore, by the constant sign theorem, $f(x) < 0$ on $(-\infty, 0)$ and $f(x) > 0$ on $(0, \infty)$.

   (c) Recall that one way to show a function is increasing is to show where its derivative is positive. Similarly, to show where the function is decreasing, it is enough to show where its derivative is negative. Thus, we can use the constant sign theorem but applied to f'. We recall $f'(x) = 0$ only $x = -1$ and note that f' is continuous everywhere. Therefore, f' is nonzero and continuous only on the intervals $(-\infty, -1)$ and $(-1, \infty)$, which are thus our intervals of interest. We find $f'(-2) = 1 > 0$ and $f'(0) = 1 > 0$. Using the constant sign theorem, we may then conclude that $f'(x) > 0$ on both $(-\infty, -1)$ and $(-1, \infty)$.

   (d) Recall that it is sufficient to show $f''(x) > 0$ on some interval to conclude f is concave up here. Moreover, showing $f''(x) < 0$ on some interval is enough to conclude f is concave down there. Therefore, using the constant sign theorem and finding where $f''(x) = 0$ can be used to solve this problem. We find $f''(x) = 2x + 2 = 2(x+1)$, and so $f''(x) = 0$ only at $x = -1$. Therefore, f'' is continuous and nonzero on the intervals $(-\infty, -1)$ and $(-1, \infty)$. We find $f''(-2) = -2$ and $f''(0) = 2$. Therefore, by the constant sign theorem, f is concave down on $(-\infty, -1)$ and concave up on $(-1, \infty)$.

   (e) We recall that in order for a point x to be an inflection point, $f''(x)$ must be zero; however, that is not to say that if $f''(x) = 0$ that x is an inflection point. But we have just shown $f''(x) = 0$ only $x = -1$ and so $x = -1$ is the only candidate for being an inflection point. Moreover, we found that f is concave down to the left of $x = -1$ and concave up to the right of $x = -1$, which is sufficient to conclude $x = -1$ is an inflection point of f. That is, $(-1, f(-1))$ is a point of inflection for f.

   (f) We recall that in order for x to be a relative extremum, $f'(x) = 0$ or $f'(x)$ must not exist while x is in the domain of f. Thus, since we have shown that $f'(x) = 0$ only at $x = -1$, $x = -1$ is the only candidate for being a relative extremum. However, we have shown that $x = -1$ was an inflection point and therefore f hasn't any relative extrema.

Problem 2. If they exist, find the absolute extrema of the following functions on the indicated intervals.

1. $f(x) = -e^{-x}(x+2)$ on $[-3, 3]$

Solution 2. For these problems, our plan is to find the derivative of the function, find the critical points, then compare the values of the function with the function at the endpoints or appropriate limits.

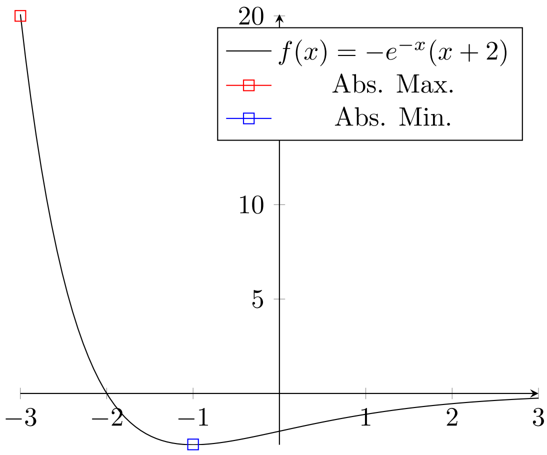
1. Our plan We begin by finding the derivative of f. First note $f(x) = -xe^{-x} - 2e^{-x}$. We find

$$f'(x) = -e^{-x} + xe^{-x} + 2e^{-x} = xe^{-x} + e^{-x} = (x+1)e^{-x}.$$

Thus, since $e^{-x} > 0$ (i.e., e^{-x} is never zero), we find that only $x = -1$ is a critical point of f. Moreover, -1 is in fact in the interval $[-3, 3]$, so we must consider the function at -1. Furthermore, since we are considering the interval $[-3, 3]$ which contains its endpoints -3 and 3, we must consider the function at these points as wel.. We construct the following table

Value	Points
0	$f(-1) = -e \sim -2.7$
-3	$f(-3) = e^3 \sim 20.0$
3	$f(3) = -5e^{-3} \sim 0.3$

which shows us that, since $f(-3)$ is the largest value in this table, $f(-3)$ is the absolute maximum for f on $[-3, 3]$. Moreover, since $f(-1)$ is the least number in the table, $f(-1)$ is thus the absolute minimum for f on $[-3, 3]$. The following is a plot of this situation where the max and min are indicate by the red and blue boxes.



Problem 3. Sketch a graph of the function $f(x) = \frac{x}{x+1}$.

Solution 3. Let us work out what we need to first and then organize everything into a table.

We note that f is not defined only at $x = -1$ and is in fact continuous and defined everywhere else Next, note that f is not symmetric (as you can check). Now, $x = -1$ is a vertical asymptote since this is where f "blows" up; i.e., this is where f may go to positive or negative infinity. Let's determine the behavior of f at this point.

We shall make use of the constant sign theorem. First note that $f(x) = 0$ only at $x = 0$ since this is where the numerator is zero. Thus, our intervals where f is both continuous and nonzero are the intervals $(-\infty, -1), (-1, 0)$, and $(0, \infty)$–we are including $(0, \infty)$ just to gain extra information about the sign of f; i.e., we need only care about the first two intervals since only these will give us information about how the vertical asymptote $x = -1$ behaves. Let's test a point from each interval.

Interval	Test Point	Value	Sign of function on Interval
$(-\infty, -1)$	-2	$f(-2) = 2$	positive
$(-1, 0)$	$-\frac{1}{2}$	$f(-\frac{1}{2}) = -1$	negative
$(0, \infty)$	1	$f(1) = \frac{1}{2}$	positive

From this we conclude that $f(x)$ is positive just left of -1 and negative just right of -1. It follows that as we take $x \to -1$ from the left that, despite f "blowing up", $f(x)$ will always be positive;

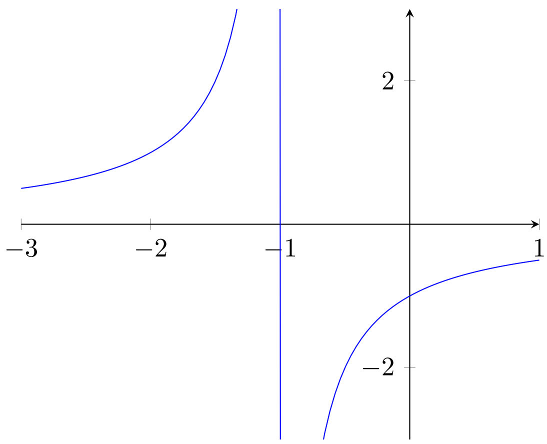
$$\lim_{x \to -1^-} f(x) = +\infty.$$

Similarly, as $x \to -1$ from the right, $f(x)$ is always negative, from which we conclude

$$\lim_{x \to -1^+} f(x) = -\infty.$$

Therefore, the following graph depicts what the function approximately looks like, where the vertical line at $x = -1$ indicates a vertical asymptote.

Sketch with vertical asymptote



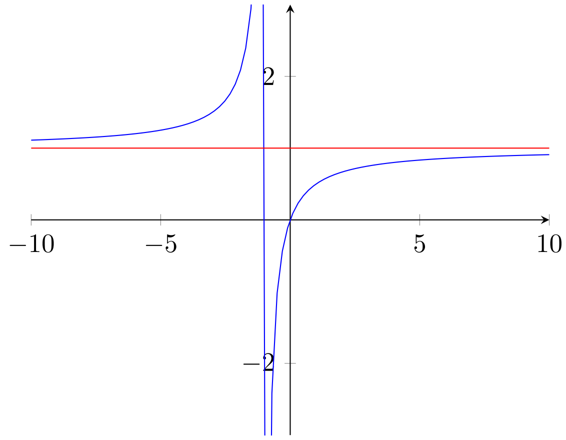
We see that as we approach $x = -1$ from the right, the plot goes off towards $-\infty$. Similarly, as we approach $x = 1$ from the left, the plot goes off towards $+\infty$. Note that this graph is really just to help aid in what one should have in their mind during this process–you will probably not have such a sketch yet until after you do more work.

Now we need to check for the horizontal asymptotes of the function; viz., we need to test the behavior of f as $x \to -\infty$ and as $x \to +\infty$. We find

Infinity	Limit	Asymptote?
$-\infty$	$\lim_{x \to -\infty} \frac{x}{x+1} = 1$	Yes
$+\infty$	$\lim_{x \to +\infty} \frac{x}{x+1} = 1$	Yes

What this table says is that as x approaches $-\infty$ or $+\infty$, then $f(x)$ will approach the value 1 and therefore, since 1 is a finite number, we may conclude $y = 1$ is the horizontal asymptote of f. Thus, we should expect that as $x \to -\infty$, f should get as arbitrarily close to $y = 1$ as we like. This is similarly true for the case that $x \to +\infty$. We thus have painted to following picture, where the vertical blue line is still the vertical asymptote at $x = -1$ and the horizontal red line $y = 1$ is the horizontal asymptote.

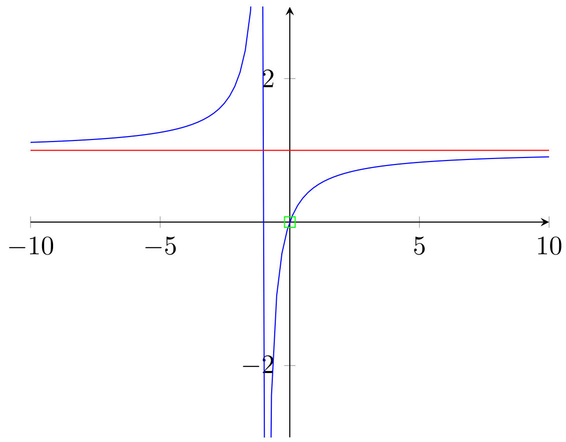
Sketch with vertical/horizontal asymptotes



The same note from above applies here–you will probably not have such a sketch yet.

We now find the x and y intercepts. The x intercept occurs when f intersects the x-axis; i.e., when $f(x) = 0$, which we found to be $x = 0$. The y intercept occurs when f intersects the y-axis; i.e., when $x = 0$, which gives a y intercept of $y = f(0) = 0$. Let us label these points on the previous graph by a green box.

Sketch with vertical/horizontal asymptotes, x, y intercepts



This is about all the information we can get from f. We now deal with f', which turns out to be

$$f'(x) = \frac{1}{(x+1)^2}.$$

We must find the critical points of f, which are when $f'(x) = 0$ or $f'(x)$ DNE and x is in the domain of f. Note that $f'(x)$ is never zero and that $f'(x)$ is only undefined at $x = -1$. However, $x = -1$ is not in the domain of f and so is not a critical point. But, we still must consider this point when using the constant sign theorem.

We now find where f is increasing and where it is decreasing. To do so, we find where $f'(x) > 0$ and where $f'(x) < 0$ by using the constant sign theorem. We note that $f'(x)$ is never zero and $f'(x)$ is only discontinuous/not defined at $x = -1$. Therefore, the intervals of interest are $(-\infty, -1)$ and $(-1, \infty)$. Let's test a point from each interval.

Interval	Test Point	Value	Sign of derivative on Interval
$(-\infty, -1)$	-2	$f'(-2) = 1$	positive
$(-1, \infty)$	0	$f(0) = 1$	positive

which tells us that $f'(x) > 0$ on both $(\infty, -1)$ and $(-1, \infty)$. This is enough to conclude that f is always increasing. We note that since we had no critical points that f has no relative extrema.

We now move on to f'', which we find to be

$$f''(x) = -\frac{2}{x+1}^{3}.$$

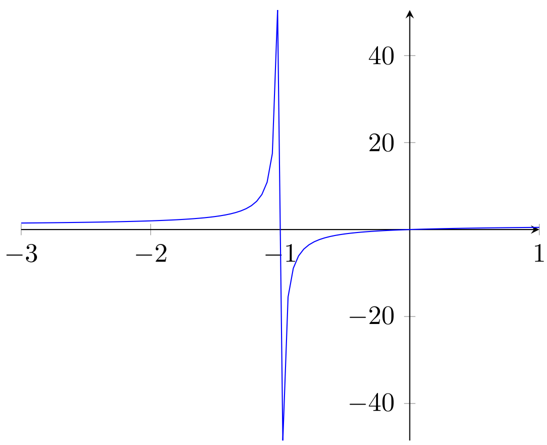
We shall now use the second derivative to determine where the function f is concave up and concave down. Recall that to do this, we need to determine where $f''(x) > 0$ for concave up and $f''(x) < 0$ for concave down. It follows that we need to find where $f''(x) = 0$ or is discontinuous or is not defined. Note first that $f''(x)$ is never zero for the same reason as why f' was never zero. Moreover, $f''(x)$ is only not defined/discontinuous at $x = -1$ since this is where we would "divide by zero." Thus, in accordance with the constant sign theorem, the interavls of interest are $(-\infty, -1)$ and $(-1, \infty)$. Let's test a point from each interval.

Interval	Test Point	Value	Sign of 2nd derivative on Interval
$(-\infty, -1)$	-2	$f''(-2) = 2$	positive
$(-1, \infty)$	0	$f(0) = -2$	negative

which tells us that $f''(x) > 0$ on $(-\infty, -1)$ and $f''(x) < 0$ on $(-1, \infty)$. That is, f is concave up on $(-\infty, -1)$ since this is where $f''(x) > 0$, and f is concave down on $(-1, \infty)$ since this is where $f''(x) < 0$.

Now, using the information just acquired about f' and f'', we can determine where f is increasing/decreasing and where it is concave up/down. Using this and perhaps a few values of f, we may now sketch an approximate plot of the function. Compare this information to the following graph that I plotted, which happens to be the exact plot of f (ignore the vertical line, this is just an bug in the program used for plotting).

Graph of $f(x) = \frac{x}{x+1}$



Problem 4. Find the following limits.

1. $\lim_{x\to\infty} \sqrt{1+e^{-x}}$
2. $\lim_{x\to-\infty} \frac{e^x+x}{2x+1}$.

Solution 4.

1. A usual trick for this problem is to realize that we may take the limit inside the square root because the square root function is continuous; i.e.,

$$\lim_{x\to\infty} \sqrt{1+e^{-x}} = \sqrt{\lim_{x\to\infty} 1+e^{-x}}.$$

Thus, we need only consider the computation

$$\lim_{x\to\infty} 1+e^{-x}.$$

Recall that the exponential e^x "behaves" like the "largest degree polynomial". Thus, we note that $e^{-x} = \frac{1}{e^x}$ must go to 0 as $x \to \infty$ since as $x \to \infty$, e^x will become arbitrarily large. That is

$$\lim_{x\to\infty} 1+e^{-x} = 0,$$

from which we find

$$\lim_{x\to\infty} \sqrt{1+e^{-x}} = \sqrt{\lim_{x\to\infty} 1+e^{-x}} = \sqrt{1+0} = 1.$$

2. For this problem, we recall the following "trick"

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(-x),$$

so long as these limits are defined. Therefore, in order to find the desired limit, we can transform the problem into

$$\lim_{x\to-\infty} \frac{e^x+x}{2x+1} = \lim_{x\to\infty} \frac{e^{-x}-x}{-2x+1},$$

where we have just simply replaced all x's with $-x$'s in the right hand side. Thus, we find

$$\lim_{x\to\infty} \frac{e^{-x}-x}{-2x+1} = \lim_{x\to\infty} \frac{x}{x}\left(\frac{\frac{e^{-x}}{x}-1}{-2+\frac{1}{x}}\right) = \lim_{x\to\infty}\left(\frac{\frac{e^{-x}}{x}-1}{-2+\frac{1}{x}}\right) = \frac{\lim_{x\to\infty}(\frac{1}{xe^x}-1)}{\lim_{x\to\infty}(-2+\frac{1}{x})} = \frac{-1}{-2} = \frac{1}{2}.$$

Problem 5. Find inflection points for the following functions

1. $f(x) = x^4$.
2. $g(x) = \frac{2}{x^2} + \frac{1}{x}$.

Solution 5. Recall that in order for a point x to be an inflection point of some function F, the point x must meet the following properties.

i. On some interval (a, x), F'' must be positive (resp. negative).

ii. On some interval (x, b), F'' must be negative (resp. positive).

iii. $F''(x) = 0$.

That is, F must be concave up (resp. down) just to the left of x and concave down (resp. up) just to the right of x (as well as have $F''(x) = 0$) in order for x to be an inflection point. Note that this does not say that if $F''(x) = 0$, then x is an inflection point–properties i. and ii. must also be met. Thus, the general scheme is to find F'', solve x in $F''(x) = 0$, and then test for conditions i. and ii..

1. We must first find f''. Firstly,

$$f'(x) = 4x^3,$$

from which

$$f''(x) = 12x^2.$$

It follows that $f''(x) = 0$ only at $x = 0$ and therefore $x = 0$ is the only candidate for being an inflection point of f. We must now use the constant sign theorem. We note that f'' is nonzero and continuous on the two intervals $(-\infty, 0)$ and $(0, \infty)$. We test points in these intervals.

Test Point	Value	Concavity
-1	$f''(-1) = 12 > 0$	Up
1	$f''(1) = 12 > 0$	Up

from which we conclude that since conditions i. and ii.. are not met that $x = 0$ is **not** an inflection point since f'' "goes" form positive to positive as we "cross" $x = 0$; i.e., to the left of $x = 0$, f is concave up *and* to the right of $x = 0$, f is still concave up. That is, f does not have any inflection points.

As an important aside, this example shows that the second derivative of a function may be zero despite the function not having any inflection points.

2. Again, find g''. Firstly,

$$g'(x) = -\frac{4}{x^3} - \frac{1}{x^2}$$

from which

$$g''(x) = \frac{12}{x^4} + \frac{2}{x^3}.$$

We wish to find when $g''(x) = 0$, which is found by multiplying through by x^4. So

$$\frac{12}{x^4} + \frac{2}{x^3} = 0$$

gives

$$x^4(\frac{12}{x^4} + \frac{2}{x^3}) = 12 + 2x = 0.$$

Thus $x = -6$ is the only point for which $g''(x) = 0$. Therefore, $x = -6$ is the only candidate for an inflection point of g. We now need to test conditions i. and ii.. We do so by the constant sign theorem. First note g'' is only discontinuous at $x = 0$ and so g'' is continuous and nonzero on the intervals $(-\infty, -6), (-6, 0)$, and $(0, \infty)$.

Interval	Test Point	Value	Concavity
$(-\infty, -6)$	-7	$g''(-7) = -\frac{2}{2401}$	down
$(-6, 0)$	-5	$g''(-5) = \frac{2}{625}$	up
$(0, \infty)$	1	$g''(1) = 14$	up

from which we conclude that conditions i. and ii.. are met for $x = -6$ and so $x = -6$ is an inflection point. That is, we see that just to the left of $x = -6$ that g is concave down and just to the right of $x = -6$ that g is concave up, which is enough to conclude $x = -6$ is an inflection point (since $g''(-6) = 0$). The following graph depicts what this inflection point looks like.