The Fundamental Theorem of Calculus

The main goal of this section is to state and use what is known as the fundamental theorem of calculus (FTOC).

Theorem. *Suppose f is continuous on $[a,b]$. Then*

$$\int_a^b f(x)dx = F(x)\big|_a^b = F(b) - F(a),$$

where F is any antiderivative of f.

Thus, we see that in order to compute the definite integral $\int_a^b f(x)dx$, we need only find an antiderivative of f. The general procedure is to compute the indefinite integral $\int f(x)dx$, and then set the integration constant to $C = 0$ to obtain an antiderivative.

Typical examples

Example 1. *Compute $\int_0^1 x + 1dx$.*

We do the following computation

$$\int_0^1 x + 1dx = \frac{x^2}{2} + x\big|_0^1 = \frac{1}{2} + 1 - 0 = \frac{3}{2}.$$

△

Example 2. *Compute $\int_1^b \frac{1}{2x}dx$*

We find that

$$\int_1^b \frac{1}{2x}dx = \frac{1}{2}\int_1^b \frac{1}{x}dx = \frac{1}{2}\left(\ln|x|\big|_1^b\right) = \frac{1}{2}\left(\ln|b| - \ln|1|\right) = \frac{1}{2}\ln|b|.$$

△

Example 3. *Compute $\int_{-2}^{-1} \frac{1}{2x}dx$*

We find that

$$\int_{-2}^{-1} \frac{1}{2x}dx = \frac{1}{2}\left(\ln|x|\big|_{-2}^{-1}\right) = \frac{1}{2}\ln 2.$$

△

Examples with u-substitution

Example 4. *Compute $\int_0^1 \frac{1}{(1+x)^2}dx$*

We need to do a u-sub. We let $u = 1+x$, and so $du = dx$. Note that the bounds are now $u = 1+0 = 1$ and $u = 1+1 = 2$. Then

$$\int_0^1 \frac{1}{(1+x)^2}dx = \int_1^2 \frac{1}{u^2}du = -u^{-1}\big|_1^2 = -2^{-1} + 1^{-1} = \frac{1}{2}.$$

△

Example 5. *Compute $\int_2^B (x+3)^5\, dx$.* △

Area between two curves

Theorem. *Suppose f is nonnegative on $[a,b]$. Then $\int_a^b f(x)dx$ gives the area of the region enclosed by the graph of f, the x-axis, and the lines $x=a$ and $x=b$.*

Example 6. *Compute the area enclosed by the curve $f(x)=e^x$, the x-axis, and $x=1$ and $x=2$.*

Since $e^x>0$, we wish to find

$$\int_1^2 e^x dx = e^x\big|_1^2 = e^2 - e^1.$$

△

Theorem. *Suppose $f(x)\geq g(x)$ on $[a,b]$. Then $\int_a^b f(x)-g(x)dx$ gives the area of the region enclosed by the graphs of f and g, and the lines $x=a$ and $x=b$.*

Theorem. *Suppose f is continuous on $[a,b]$ and $a<c<b$. Then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

We will be concerned only with finding the area in the bounded region bounded by two curves. Given two continuous functions f and g, to find the area between these two curves, we have the following general scheme.

1. Find all x such that $f(x)=g(x)$.
2. Find all bounded intervals where $f(x)\geq g(x)$ and find all bounded intervals where $g(x)\geq f(x)$ (e.g., use the constant sign theorem).
3. Let $[a,b]$ be a general such interval. If $f(x)\geq g(x)$ on $[a,b]$, compute $\int_a^b f(x)-g(x)dx$. If $g(x)\geq f(x)$ on $[a,b]$, compute $\int_a^b g(x)-f(x)dx$.
4. Add up all such integrals.

Example 7. *Find the area enclosed by the curves $f(x)=e^x$, $g(x)=x$, $x=1$ and $x=2$.*

We note that $e^x-x\geq 0$ on $[1,2]$ and so we wish to compute

$$\int_1^2 e^x - xdx = e^x - \frac{x^2}{2}\Big|_1^2.$$

△

Example 8. *Find the area enclosed by the curves $f(x)=x^3$ and $g(x)=x$.*

1. $f(x)=g(x)$ when $x^3=x$; i.e., when $x=\pm 1$ or $x=0$.
2. By the following table

x	-1	-.5	0	.5	1
$f(x)-g(x)$	0	0.375	0	-0.375	0
sign	0	$>$	0	$<$	0

   we conclude that $f(x)-g(x)>0$, and so $f(x)>g(x)$, on $(-1,0)$ and $f(x)-g(x)<0$, and so $f(x)<g(x)$, on $(0,1)$. These are the only bounded intervals of interest.

3. The area between the curves is thus given by

$$\int_{-1}^0 f(x)-g(x)dx + \int_0^1 g(x)-f(x)dx = \int_{-1}^0 x^3 - xdx + \int_0^1 x - x^3dx = \frac{1}{4}+\frac{1}{4}=\frac{1}{2}.$$

△

Example 9. *Find the area enclosed by the curve $y = x^2 - 9$, the x-axis, and the lines $x = -7$ and $x = 0$.*
Here, our two curves are $f(x) = x^2 - 9$ and $g(x) = 0$.

1. $f(x) = g(x)$ when $x^2 - 9 = 0$; i.e., when $x = \pm 3$.
2. By the following table

x	-7	-4	-3	-1	0
$f(x)$	40	7	0	-7	-9
sign		$>$		$<$	

we conclude that $f(x) - g(x) > 0$, and so $f(x) > g(x)$, on $(-7, -3)$ and $f(x) - g(x) < 0$, and so $f(x) < g(x)$, on $(-3, 0)$. These are the only bounded intervals of interest.

3. The area between the curves is thus given by

$$\int_{-7}^{-3} f(x) - g(x)dx + \int_{-3}^{0} g(x) - f(x)dx = \int_{-7}^{-3} x^2 - 9dx + \int_{-3}^{0} -9 + x^2 dx = \frac{208}{3} + 18.$$

△

Example 10. *Find the area enclosed by the curves $f(x) = \frac{2}{x}$ and $g(x) = 1$ when $1 \leq x \leq 4$.*

1. $f(x) = g(x)$ when $\frac{2}{x} = 1$; i.e., when $x = 2$.
2. By the following table

x	1	1.5	2	3	4
$f(x) - g(x)$	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{2}$
sign		$>$		$<$	

we conclude that $f(x) - g(x) > 0$, and so $f(x) > g(x)$, on $(1, 2)$ and $f(x) - g(x) < 0$, and so $f(x) < g(x)$, on $(2, 4)$. These are the only bounded intervals of interest.

3. The area between the curves is thus given by

$$\int_{1}^{2} f(x) - g(x)dx + \int_{2}^{4} g(x) - f(x)dx = \int_{1}^{2} \frac{2}{x} - 1dx + \int_{2}^{4} 1 - \frac{2}{x}dx = \ln(4) - 1 + \log(4) - 2 = 2\ln 4 - 3.$$

△

Example 11. Graph example done in class. △