1 First Derivative Test

In this section we define and demonstrate how to find what are called relative extrema. We shall do this by example because, as is often the case, mathematical rigor will obfuscate the issue.

So, let $f(x) = x^2$. If you plot the graph of this function, it is not surprising that we would say $f(0) = 0$ is the minimum of this function. However, what if we graphed the function $g(x) = x^3 - x$ (plot this using W—A)? We would find that this function is unbounded in both the positive and negative directions. However, there are what appear to be a *local* maximum and a *local* minimum at $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$, respectively. We shall call these *local* extrema *relative extrema.*

To give proper definitions, we go back to $f(x) = x^2$. We shall demonstrate a method in finding its relative (in this case, absolute) extrema using tables. It turns out that its derivative, $f'(x) = 2x$, plays a crucial role.

After finding $f'(x) = 2x$, we find the critical points of f. Note that $x = 0$ is the only critical point. We now choose two points close to $x = 0$, one to left and one to the right. Let's choose the points $x = -1$ and $x = 1$ (how we choose points will be important for more general problems). It will be important to use the values of f' at these points: $f'(-1) = -2$ and $f'(1) = 2$. We begin tabulating this data:

	-1	0	1
$f'(x)$	$-$	0	$+$

where we have used a $-$ to indicate f' is negative at $x = -1$, and a $+$ to indicate f' is positive at $x = 1$.

Now, by the constant sign theorem, we may actually conclude that $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, +\infty)$ (why?). Next, recalling that $f'(x) < 0$ on an interval implies f is decreasing on that interval, and $f'(x) > 0$ on an interval implies f is increasing on that interval, we conclude that f is decreasing on $(-\infty, 0)$ and increasing on $(0, +\infty)$. We indicate this by $\nearrow$ to mean increasing and $\searrow$ to mean decreasing:

	-1	0	1
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$		$\nearrow$

We may conclude that f goes from decreasing to increases as x goes from -1 to 1. Moreover, this "switch" happens at $x = 0$. When such a switch happens at a point $x = c$, we call this switch a relative extremum. That is, for $f(x) = x^2$, $x = 0$ is a relative extremum, and in fact will be called a relative minimum. We complete the table by noting this:

	-1	0	1
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	minimum	$\nearrow$

The above demonstrates what is called the **first derivative test**. In a word, it consists of finding critical points, choosing points between critical points, testing the derivative at these points, and then analyzing using the constant sign theorem whether or not we get a relative extremum. We list out the relevant tables. Let f be a function and f' its derivate. Let c be a critical point of f with $f'(c) = 0$ and u and v two points such that $u < c < v$ and no other critical points lie in this interval. Then the first derivative test states:

	u	c	v
$f'(x)$	$+$	0	$+$
$f(x)$	$\nearrow$	neither	$\nearrow$

	u	c	v
$f'(x)$	$+$	0	-
$f(x)$	$\nearrow$	maximum	$\searrow$

	u	c	v
$f'(x)$	-	0	$+$
$f(x)$	$\searrow$	minimum	$\nearrow$

	u	c	v
$f'(x)$	-	0	-
$f(x)$	$\searrow$	neither	$\searrow$

where the conclusion is in the last row's entry under "c" (i.e., the "neither", "minimum", or "maximum".)

Let's see an example that demonstrates the general scheme of these types of problems.

Example. *Let* $f(x) = \frac{x^4}{4} + \frac{x^3}{3} + 1$*, find where* f *is increasing and decreasing, and find all absolute extrema of* f*.*

1. We first find f':

$$f'(x) = x^3 + x^2 = x^2(x+1).$$

2. We find the critical points, with special attention to where $f'(x) = 0$. But $x^2(x+1) = 0$ only at $x = 0$ and $x = -1$. Thus the critical points are $x = 0$ and $x = -1$.

3. We begin tabulating.

	-1	0
$f'(x)$	0	0
$f(x)$		

If $f'(c)$ did not exist, we would write DNE in place of 0. Note that this is only important for finding where f is increasing or decreasing.

4. Based off the critical points, we choose test points from each maximal interval not containing the critical points. These intervals are $(-\infty, -1)$, $(-1, 0)$ and $(0, +\infty)$. Let's choose the points -2, -0.5, and 1 and tabulate:

	-2	-1	-0.5	0	1
$f'(x)$		0		0	
$f(x)$					

5. Here we test f' at each of the test points and tabulate. Note $f'(-2) = -4 < 0$, $f'(-0.5) = \frac{1}{8} > 0$, and $f'(1) = 2 > 0$. Tabulating, we get:

	-2	-1	-0.5	0	1
$f'(x)$	$-$	0	$+$	0	$+$
$f(x)$					

6. Use the constant sign theorem to conclude that f is decreasing on $(-\infty, -1)$, and increasing on $(-1, 0) \cup (0, +\infty)$. Tabulating,

	-2	-1	-0.5	0	1
$f'(x)$	$-$	0	$+$	0	$+$
$f(x)$	$\searrow$		$\nearrow$		$\nearrow$

7. Lastly, we use the second derivative test to conclude $f(-1)$ is a relative minimum and $f(0)$ is neither. Tabulating, we get:

	-2	-1	-0.5	0	1
$f'(x)$	$-$	0	$+$	0	$+$
$f(x)$	$\searrow$	minimum	$\nearrow$	neither	$\nearrow$

2 Second Derivative Test

In this section, we learn another test for finding relative extrema. But first, we need some notation. For the *second derivative*, that is, the derivative of the derivative, of a function f, we use the notation:

$$f''(x) = \frac{d^2}{dx^2} f(x) = \frac{d}{dx} f'(x).$$

To discuss the second derivative test, we will need to make use of a definition called *concavity.*

1. If $f''(x) > 0$ on (a, b), then f is concave up on (a, b). Denote this by $\cup$.
2. If $f''(x) < 0$ on (a, b), then f is concave down on (a, b). Denote this by $\cap$.

In practice, when finding where f is concave up or down, we will make use of the constant sign theorem. The process is basically identical to that of finding where f was increasing or decreasing.

Before we see example, we define *inflection points*. First, let $f'(c) = 0$ and (a, c) and (c, b) be such that these intervals contain no other critical points. Then, *If $f''(x) > 0$ on (a, c) and $f''(x) < 0$ on (c, b), or vice verse, then $(c, f(c))$ is an inflection point.*

We shall do this by example.

Example. *Let $f(x) = \frac{x^4}{12} + \frac{x^3}{6} + 2$, and find where f is concave up and where f is concave down. Find any inflection points.*

1. We first find the second derivative of f: $f''(x) = x^2 + x = x(x + 1)$.
2. To use the constant sign theorem on f'', we need to find where f'' is nonzero and continuous. Note that this is the same as finding the critical points of f'. Note that we want to use the constant sign theorem on f'' to determine where $f'' > 0$ and where $f'' < 0$, and hence to find where f is concave up or down, respectively. But, f'' is a polynomial and hence continuous everywhere. Moreover, by solving $f''(x) = x(x + 1) = 0$, we find that $f''(x) = 0$ at $x = 0$ and $x = -1$.
3. We tabulate:

	-1	0
$f''(x)$	0	0
$f(x)$		

4. Based where $f''(x) = 0$ and discontinuous (which is nowhere), we choose test points from each maximal interval not containing these points. These intervals are $(-\infty, -1)$, $(-1, 0)$ and $(0, +\infty)$. Let's choose the points -2, -0.5, and 1 and tabulate:

	-2	-1	-0.5	0	1
$f''(x)$		0		0	
$f(x)$					

5. Here we test f'' at each point: $f''(-2) = 2 > 0$, $f''(-0.5) = -\frac{1}{4}$, and $f''(1) + 2 > 0$. Tabulating, we get:

	-2	-1	-0.5	0	1
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$					

6. We now tabulate by applying the definition of concavity:

	-2	-1	-0.5	0	1
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$		$\cap$		$\cup$

   where $\cap$ indicates concave down and $\cup$ indicates concave up.

7. We conclude, by the constant sign theorem, that f is concave up on $(-\infty, -1) \cup (1, +\infty)$, and concave down on $(-1, 0)$. Moreover, both $(0, f(0))$ and $(-1, f(-1))$ are inflection points.

We conclude by stating and applying the second derivative test:

Theorem. *Suppose $f'(c) = 0$. Then*

1. *If $f''(c) > 0$ ($f''(c) < 0$), then $f(c)$ is a relative minimum (maximum).*

2. *If $f''(c) = 0$, we cannot conclude anything.*

Note that if $f''(c) = 0$, then we will need to use the first derivative test if we determine $(c, f(c))$ is not an inflection point.

Example. *Let $f(x) = \frac{x^4}{12} - \frac{x^3}{6} + 2$ and find the relative extrema of f, including any inflection points.*

1. We first find the derivatives of f: $f'(x) = \frac{x^3}{3} - \frac{x^2}{2}$ and $f''(x) = x^2 - x = x(x-1)$.

2. We find the critical points of f by solving $f'(x) = 0$. This occurs at $x = 0$ and $x = \frac{3}{2}$.

3. To find inflection points, we will need to know where $f'' > 0$ and $f'' < 0$. Thus, we need to apply the constant sign theorem, and so we need to know where $f''(x) = 0$. This occurs at $x = 0$ and $x = 1$.

4. We apply the second derivative test at the critical points $x = 0$ and $x = \frac{3}{2}$. We find $f''(0) = 0$ and thus we cannot conclude anything. If $x = 0$ is not an inflection point, we will need to use the second derivative test.

   However, $f''(\frac{3}{2}) = \frac{3}{4} > 0$, and therefore $f(\frac{3}{2})$ is a relative minimum.

5. Filling out the table, we get

	-1	0	0.5	1	2
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$		$\cap$		$\cup$

   Thus 0 and 1 are the x-coordinates of inflection points.

Horizontal Asymptotes

In this section, we explore the concept of horizontal asymptotes. Horizontal asymptotes are much like vertical asymptotes in that they are determined by finding a limit. However, while vertical asymptotes are found by taking the limit as x approaches some finite number, horizontal asymptotes are found by taking the limit as $x \to +\infty$ or $x \to -\infty$. So, what do these limits actually mean? Heuristically, they should be understood as follows. If f is a function, then the limit

$$\lim_{x \to +\infty} f(x)$$

is to mean the value $f(x)$ *tends* to as x becomes arbitrarily large in the positive direction. Similarly, the limit

$$\lim_{x \to -\infty} f(x)$$

should be understood as the value $f(x)$ *tends* to as x becomes arbitrarily large in the negative direction.

Example. Let $f(x) = x^2$ and $g(x) = \frac{1}{x}$. Then,

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^2 = +\infty$$

because as x becomes arbitrarily large in the positive direction, so does x^2. However,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^2 = +\infty$$

as well because despite x becoming arbitrarily large in the negative direction, x^2 remains positive.

Next,

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} \frac{1}{x} = 0$$

because as x becomes arbitrarily large (in either direction), $\frac{1}{x}$ becomes arbitrarily small. Similarly,

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{1}{x} = 0$$

for the same reason.

We have the general facts.

Theorem.

1.

$$\lim_{x\to+\infty} x^p = +\infty$$

when $p > 0$*.*

2.

$$\lim_{x\to-\infty} x^p = +\infty$$

when $p > 0$ *is an even integer*

3.

$$\lim_{x\to-\infty} x^p = -\infty$$

when $p > 0$ *is an odd integer.*

4.

$$\lim_{x\to+\infty} x^p = 0$$

when $p < 0$*.*

5.

$$\lim_{x\to-\infty} x^p = 0$$

when $p < 0$ *is an integer.*

We may now define horizontal asymptotes.

Definition (Horizontal Asymptote)**.** Let f be a function. Then, if either of the limits

$$\lim_{x\to+\infty} f(x) \qquad \text{and} \qquad \lim_{x\to-\infty} f(x)$$

exist and are finite, then the limit is called a horizontal asymptote.

So, finding horizontal asymptotes essentially answers the question: How does the function behave "at" infinity? Let's see some examples.

Example. In this example, we will compute several horizontal asymptotes. Also, as a corollary, we will state a general rule for finding horizontal asymptotes as we go.

(a) *If they exist, find the horizontal asymptotes of* $f(x) = \frac{x^2+x}{x^3+2x+1}$

We are basically being asked to compute the limits

$$\lim_{x\to+\infty} f(x) \qquad \text{and} \qquad \lim_{x\to-\infty} f(x).$$

The trick is to realize that since the degree of the numerator is less than the degree of the denominator, the denominator is going to dominate the expression as $x \to \pm\infty$. That is, as x becomes arbitrarily large, $\frac{x^2+x}{x^3+2x+1}$ tends to 0 since the x^3 dominates everything in this expression. Thus

$$\lim_{x\to+\infty} f(x) = 0.$$

Next, as $x \to -\infty$, x is becoming arbitrarily large in the negative direction. Thus, again, x^3 will dominate the rational expression. Thus, again,

$$\lim_{x\to-\infty} f(x) = 0.$$

Therefore, the horizontal asymptote is $y = 0$ as $x \to \infty$ and $x \to -\infty$.

This allows us to state a general rule: **If the degree of the denominator is greater than the numerator, the horizontal asymptotes exists and are equal to 0.**

Lastly, we can do the following shortcut by filling in the table:

	Dominating Term	$x \to +\infty$	$x \to -\infty$
Numerator	x^2		
Denominator	x^3		
Leftover	x^{-1}	0	0

(b) *If they exist, find the horizontal asymptotes of* $f(x) = \frac{2x^5+2x+2}{x^2+2}$

We fill out the following table:

	Dominating Term	$x \to +\infty$	$x \to -\infty$
Numerator	$2x^5$		
Denominator	x^2		
Leftover	$2x^3$	$+\infty$	$-\infty$

This indicates that

$$\lim_{x\to+\infty} f(x) = +\infty \qquad \lim_{x\to-\infty} f(x) = -\infty,$$

and so f does not have any horizontal asymptotes.

(c) *If they exist, find the horizontal asymptotes of* $f(x) = \frac{-3x^7+2x+2}{x^7+2}$

We fill out the following table:

	Dominating Term	$x \to +\infty$	$x \to -\infty$
Numerator	$-3x^7$		
Denominator	x^7		
Leftover	-3	-3	-3

This indicates that

$$\lim_{x\to+\infty} f(x) = -3 \qquad \lim_{x\to-\infty} f(x) = -3,$$

and so f has a horizontal asymptote of $y = -3$.

The last type of problem we need to consider are limits that involve the exponential e^x. We state a heuristic theorem.

Theorem.

1.

$$\lim_{x\to+\infty} e^x = +\infty \qquad \lim_{x\to-\infty} e^x = 0$$

2. e^x dominates polynomials as $x \to +\infty$.

3. e^{-x} is dominated by polynomials as $x \to +\infty$ (including constants).

4. e^x is dominated by polynomials as $x \to +\infty$ (including constants).

5. e^{-x} dominates polynomials as $x \to -\infty$.

Thus, when determining the "dominating term" in our tables, we may apply this theorem. For example, in $e^x + x$, e^x is the dominating term as $x \to +\infty$, yet x is the dominating term as $x \to -\infty$. However, in $e^{-x} + 3$, 3 is the dominating term as $x \to +\infty$, yet e^{-x} is the dominating term as $x \to -\infty$.

Let's see some examples with the exponential.

Example 2.0.1. Here we will use the table method as above.

1. *If they exist, find the horizontal asymptotes of* $f(x) = \frac{-3x^7+2x+2}{e^{2x}+2}$

   Now, because of the e^{2x} term in the denominator, the "dominating term" changes depending on whether $x \to +\infty$ or $x \to -\infty$. So, we make two tables, one for each limit.

   We fill out the following table:

	Dominating Term	$x \to +\infty$
Numerator	$-3x^7$	
Denominator	e^{2x}	
Leftover	$\frac{-3x^7}{e^{2x}}$	0

	Dominating Term	$x \to -\infty$
Numerator	$-3x^7$	
Denominator	2	
Leftover	$\frac{-3x^7}{2}$	$+\infty$

   This indicates that

$$\lim_{x \to +\infty} f(x) = 0 \qquad \lim_{x \to -\infty} f(x) = \infty,$$

   and so f has a horizontal asymptote of $y = 0$ as $x \to +\infty$.

2. *If they exist, find the horizontal asymptotes of* $f(x) = \frac{-e^{-2x}+4}{e^{-2x}+2}$ Now, because of the e^{-2x} term in the denominator, the "dominating term" changes depending on whether $x \to +\infty$ or $x \to -\infty$. So, we make two tables, one for each limit.

	Dominating Term	$x \to +\infty$
Numerator	4	
Denominator	2	
Leftover	2	2

	Dominating Term	$x \to -\infty$
Numerator	$-e^{-2x}$	
Denominator	e^{-2x}	
Leftover	-1	-1

   This indicates that

$$\lim_{x \to +\infty} f(x) = 0 \qquad \lim_{x \to -\infty} f(x) = -\infty,$$

   and so f has a horizontal asymptote of $y = 0$ as $x \to +\infty$.