1 Mathematical Models

In this section we will basically define a bunch of terminology, and then work out some problems that involve said terminology. But, to begin, we start off with mentioning a few words about mathematical models.

It is a rather beautiful fact that mathematics is found everywhere in application. For example, math is found in business, economics, finance, biology, medicine, or what have you. Basically, no matter where you look, you will see some sort of mathematics being used. It is also the case that a lot of people don't need to know in great detail about the mathematics used in their profession. However, in my opinion, to know gain an understanding on what math *is* and *how* it is used is a really useful tool (the same can easily be said about software engineering).

In any case, by a **mathematical model** we mean some sort of mathematical abstraction that produces some sort of predictions we find meaningful; e.g., stock price models, sabermetrics in baseball, steel integrity, or what have you. If you wish to learn more about what sort of math can be found in your field, feel free to ask and I can try to look into it.

1.1 Cost, Revenue, and Profit

We shall give definitions by example. Imagine you are running some manufacturing firm that produces steel. Say, every month it costs \$5K for upkeep of the machinery. Moreover, say every month, cost of materials amount to \$50K for x amount of steel produced.

We say the fixed \$5K a month is a **fixed cost**; i.e., a fixed cost is something independent of amount of product produced. Next, we note that we have a cost of \$50K *per* x amount of steel; i.e., the cost *depends* on the amount of product produced. In this case, we say the \$50K is the **variable cost**. Lastly, the sum total of the fixed and variable costs is called the **cost**.

Now that we have some terminology, we can move on to discussing our first example of a mathematical model. It is sometimes the case that modeling something as a linear function is useful. For us, we shall consider modeling cost, revenue, and profit linearly (and later quadratically), as these models can be useful (at least for short term models). Now, when we say we are modeling something linearly, we mean we are relating two things by a linear function. For example, in the above example, the total cost $C(x)$ was equal to $C(x) = 50x + 5$ for x amount of steal produced; i.e., the model was a linear function. We thus have the following definitions.

Definition 1 (Linear cost model)**.** Let x be the number of units of a given product produced, and m the cost of manufacturing one unit (i.e., mx gives the total cost of manufacturing x units). Then a **linear cost model** assumes m is independent of x, and so, if b is the variable cost and $C(x)$ the total cost, we have

$$\begin{aligned} \text{variable cost} &= (\text{cost per unit})\times(\text{number of of units produced}) \\ &= mx, \\ C(x) &= \text{cost} \\ &= (\text{variable cost})+(\text{fixed cost}) \\ &= mx + b. \end{aligned}$$

The next natural thing to discuss is revenue and profit. Recall that **revenue** is the amount of money made from selling a certain amount of product, and that **profit** is revenue subtracted by total cost. We state this mathematically as a definition.

Definition 2 (Revenue and profit)**.** Let x be the number of units of a given product sold, and p the price per unit sold. Then the **revenue** $R(x)$ is given by,

$$\begin{aligned} R(x) &= \text{revenue} \\ &= (\text{price per unit})\times(\text{number sold}) \\ &= px. \end{aligned}$$

Next, if we assume number of units sold equals number of units produced, and let $C(x) = mx + b$ be the cost, then the **profit** $P(x)$ is given by,

$$\begin{aligned} P(x) &= \text{profit} \\ &= (\text{revenue}) - (\text{cost}) \\ &= R(x) - C(x). \end{aligned}$$

An important concept closely related to profit is called the **break even quantity**, which is defined as the amount of units needed to be sold so that the profit is zero.

Definition 3 (Break even quantity)**.** Let P be the profit given by selling and producing x units. Then x is said to be a **break even quantity** if $P(x) = 0$.

Let's now see these definitions applied to some examples.

Example 1. *Some manufacturer has a fixed manufacturing cost of \$70 per month, material cost of \$5 per unit, and sells its product for \$10 per unit.*

(a) *Write down the linear cost, revenue, and profit functions as functions of x units made and sold.*

(b) *How many unites need to be sold to break even.*

(c) *Determine the break even revenue using the previous problem.*

(a) We identify the \$70 cost as the fixed cost, and the \$5 cost as the variable cost. Thus, the total cost is given by

$$C(x) = (\text{variable cost}) \times (\text{units made}) + (\text{fixed cost}) = 5x + 70.$$

Next, since the price is given as \$10 per unit, the revenue must be

$$R(x) = (\text{price per unit}) \times (\text{units sold}) = 10x.$$

Lastly, since the profit is given by the revenue minus the cost, we find

$$P(x) = R(x) - C(x) = 10x - 5x - 70 = 5x - 70.$$

(b) Recall that x is a break even quantity when $P(x) = R(x) - C(x) = 0$. Thus, we need to solve

$$R(x) - C(x) = 5x - 70 = 0.$$

Thus, $x = 14$ is a break even quantity, and thus 14 units need to be sold to break even.

(c) This question is just asking for the corresponding revenue when profit has broken even; i.e., what is $R(x)$ when $x = 14$. We compute

$$R(14) = 10 \cdot 14 = 140.$$

Thus, 140 is the break even revenue.

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1.2 Mathematical Models of Supply and Demand

In this section we briefly define the demand and supply equations, the equilibrium point, and see some examples.

Definition 4 (Demand equation)**.** If x is the number of units produced and sold by the entire industry during a given time period and $p = -cx + d$ for some $c > 0$ is the price of the xth unit sold, then the equation $p = -cx + d$ is the **demand equation**.

Definition 5 (Supply equation)**.** If p is a function of x and gives the price p necessary for suppliers to make available x units to the market, the equation $p = p(x)$ is called the **supply equation**.

Definition 6 (Equilibrium point)**.** If $p = p(x)$ is the supply equation and $p = -cx + d$ for some $c > 0$ is the demand equation, the **equilibrium point** is given by the point (x, p) satisfying $p(x) = -cx + d$. The x-coordinate of the equilibrium point is called the **equilibrium quantity**, and the p-coordinate of the point is called the **equilibrium price**.

Example 2. *The demand function of some brand product is given by $p = 1000 - 10x$, where p is the price per unit in dollars when x units are sold. Find the revenue function.*

This is a matter of applying the definitions of the demand function and revenue function. Recall that the revenue function is given by $R(x) = px$, where p is the price per unit in dollars, and x is the number of units sold. Thus, since the demand function gives the price in terms of x, we find,

$$R(x) = px = (1000 - 10x)x = 1000x - 10x^2.$$

△

Example 3. *Such and so is willing to supply x units when the price is $p = 3x+100$. Moreover, such and so has determined the price-demand function is $p = -x + 120$. Find the equilibrium price and quantity.*

To solve this problem, we first must extract the supply and demand equations. Looking at the definition of the supply equation, we see that x units are willing to be supplied given the price $p = 3x + 100$, and so the supply equation is $p = 3x + 100$. The demand function is given as $p = -x + 120$. Now, we must apply the definition of equilibrium price and quantity.

Thus, we find th equilibrium point by setting the supply and demand equations to each other and solve for x:

$$3x + 100 = -x + 120.$$

Solving for x yields $x = 5$. Thus, $x = 5$ is the equilibrium quantity. It follows that $p = -5 + 120 = 115$ is the equilibrium price. △

Example 4. *Such and so is willing to supply 10 units when the price per unit is \$100. If the price per unit is reduced by \$10, such and so will supply 2 fewer units Determine the supply equation for this product, assuming price p and quantity x are linearly related.*

This question boils down to solving for a linear equation provided two points. Here, the quantity x will be the "x-coordinate," and the price p will be the "y-coordinate." Our first point is given by $(x_1, p_1) = (10, 100)$. We are told that if we reduce p_1 by \$10, then x_1 will reduce by 2 units. That is to say, our other point is given by

$$(x_2, p_2) = (x_1 - 2, p_1 - 10) = (8, 90).$$

Thus, we have two points and therefore may solve for a linear equation that happens to be the supply equation.

Firstly, we find the slope:

$$m = \frac{p_2 - p_1}{x_2 - x_1} = \frac{90 - 100}{8 - 10} = 5.$$

Lastly, we apply the point slope form with $(x_1, p_1) = (10, 100)$:

$$\begin{aligned} p - p_1 &= m(x - x_1) \\ p - 100 &= 5(x - 10) \\ p &= 5x - 50 + 100 \\ p &= 5x + 50. \end{aligned}$$

Thus, the supply equation is given by $p = 5x + 50$. △

Example 5. *Research has shown that consumers are willing to purchase 10 units when the price is \$5 per unit. At \$2, consumers are willing to purchase 15 units. Find the demand equation, assuming price p and quantity x are linearly related.*

This equation is similar to the above in that we are given two points that define p as a linear equation of x, giving the demand equation. Here, quantity and price will be the x- and y- coordinates, respectively, as above. Thus, our first point is $(x_1, p_1) = (10, 5)$, and our second point is $(x_2, p_2) = (15, 2)$.

Now, we solve the problem by finding the slope of the desired line:

$$m = \frac{2-5}{15-10} = -\frac{3}{5}.$$

Now, we apply the point slope form to (x_1, p_1), and obtain

$$\begin{aligned} p - p_1 &= m(x - x_1) \\ p - 5 &= -\frac{3}{5}(x - 10) \\ p &= 11 - \frac{3x}{5}. \end{aligned}$$

Thus, $p = 11 - \frac{3x}{5}$ is our demand equation. △

1.3 Quadratic Mathematical Models

In this section we explore the idea of mathematical modeling using quadratic functions. It turns out that modeling something via a quadratic function allows one to easily find the maximum or minimum (depending on the circumstance) of whatever is being modeled.

We start off by stating a theorem about quadratic functions.

Theorem 1. *Let $q(x) = ax^2 + bx + c$, $a \neq 0$, be a quadratic function. Then q can be written as $q(x) = a(x-h)^2 + k$ for some real numbers h and k. Moreover, the point (h, k), call the* ***vertex****, can be computed by*

$$h = -\frac{b}{2a}, k = c - \frac{b^2}{4a}.$$

Lastly, if $a > 0$, then q assumes a minimum of k when $x = h$, and if $a < 0$, then q assumes a maximum of k when $x = h$.

This theorem is really useful when, say, modeling revenue by a quadratic function since we can then determine when revenue is maximized (or minimized), and what is the extremal value.

Example 6. *Find the vertex of the quadratic $2x^2 + 3x + 4$.*

This probably basically amounts to applying the definition of a vertex of a quadratic. Here, $a = 2$, $b = 3$, and $c = 4$. We wish to find (h, k) with

$$h = -\frac{b}{2a}, k = c - \frac{b^2}{4a}.$$

So, we compute

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{3}{2 \cdot 2} \\ &= -\frac{3}{4}. \end{aligned}$$

and

$$\begin{aligned} k &= c - \frac{b^2}{4a} \\ &= 4 - \frac{3^2}{4 \cdot 2} \\ &= \frac{23}{8}. \end{aligned}$$

Thus, the vertex is given by the point

$$(x, y) = (-\frac{3}{4}, \frac{23}{8}).$$

△

Example 7. *Suppose the revenue and cost functions are given by* $R(x) = -x^2 + 100x$ *and* $C(x) = 10x + 100$.

(a) *Find the maximum profit and determine how many units need to be sold to maximize profit.*

(b) *Find any break even quantities.*

(a) We first find the profit function,

$$P(x) = R(x) - C(x) = -x^2 + 100x - 10x - 100 = -x^2 + 90x - 100.$$

Thus, the profit function is a quadratic with its leading coefficient negative. This tells us that the vertex formula may be used to find the maximum value of $P(x)$ and what value of x maximizes it. Here, we use $a = -1$, $b = 90$, and $c = -100$. Then, the vertex (h, k) is given by

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{90}{2 \cdot (-1)} \\ &= 45 \end{aligned}$$

and

$$\begin{aligned} k &= c - \frac{b^2}{4a} \\ &= -100 - \frac{90^2}{4 \cdot (-1)} \\ &= 1925. \end{aligned}$$

Thus, since $a < 0$, we find that $k = 1925$ is the maximum profit value and that $P(x)$ is maximum at $x = h = 45$.

(b) Recall that x is a break even quantity when $P(x) = 0$. Thus, we must solve

$$P(x) = R(x) - C(x) = -x^2 + 90x - 100 = 0.$$

But this amounts to solving the quadratic equation as usual. After applying the quadratic formula, we find that

$$\begin{aligned} x &= -5(\sqrt{77} - 9) \\ x &= 5(9 + \sqrt{77}) \end{aligned}$$

solve the quadratic. Noting that both numbers are positive, it follows that these values of x are break even quantities.

△