1 Exponential Models

1.1 Solving an Equation with an Exponent

In this section we shall explore how to solve equations of that form $3^x = 27$. This equation is nice enough to see that $x = 3$ works since $3^3 = 3 \cdot 9 = 27$. However, we also see that $27 = 3^3$. Thus, in substituting this form of 27 into the given equation, we find that $3^x = 3^3$, suggesting that $x = 3$ is the solution. We shall explore this latter more methodological approach for solving exponential equations.

We first recall some properties of exponents that will be used throughout when solving exponential equations.

Theorem 1. *Let $a > 0$ be a real number. Then, for any x and y,*

1. *If $a \neq 1$ and $a^x = a^y$, then $x = y$.*
2. $a^x \cdot a^y = a^{x+y}$
3. $\frac{a^x}{a^y} = a^{x-y}$
4. $(a^x)^y = a^{xy}$
5. $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

We shall show by example how to solve exponential equations. However, the general scheme is as follows

1. Clear any fractions by using negative exponents.
2. If possible, rewrite everything on both sides of the equation in the same base.
3. Combine exponents in possible.
4. Compare the exponents and solve as usual.

If you have seen logarithms already, one can use logarithms in everything that follows, especially if one cannot rewrite both sides of the equation in the same base.

Example 1. *Solve for x in*

$$\frac{1}{5^{10x}} = 5^{20x+5}.$$

The first thing we need to do is clear the fraction $\frac{1}{5^{10x}}$, by which we mean, we use

$$\frac{1}{5^{10x}} = \left(5^{10x}\right)^{-1} = 5^{-10x}.$$

Thus, we hope to solve

$$5^{-10x} = 5^{20x+5}.$$

But, we may now just compare the exponents ans solve

$$-10x = 20x + 5.$$

Thus, $x = -\frac{1}{6}$ is the solution. △

Example 2. *Solve for x in*

$$36^{7x} = 6^{4x-2}.$$

The idea here is that we need to first rewrite everything in the same base if we case. That is, we need to somehow change the 36 into a power of 6. But, we know that $36 = 6^2$, and so, we have

$$36^{7x} = \left(6^2\right)^{7x} = 6^{2 \cdot 7x} = 6^{14x}.$$

Thus, we need to solve

$$6^{14x} = 6^{4x-2}.$$

But, this is a matter of solving

$$14x = 4x - 2,$$

and so, $x = -\frac{1}{5}$ is the solution. △

Example 3. *Solve for x in*

$$10^x 10^3 = 1.$$

There are two ways to do this problem. Either we can divide through by one of the factors or we can combine exponents. We shall combine exponents here. We find that

$$10^x 10^3 = 10^{x+3}.$$

Next, we need to rewrite everything in the same base. We recall that $a^0 = 1$ for all $a \neq 0$, and so

$$1 = 10^0.$$

Therefore, we wish to solve

$$10^{x+3} = 10^0;$$

i.e., we wish to solve $x + 3 = 0$. Therefore, $x = -3$ is the solution. △

Compound Interest

In this section we concern ourselves with the mathematical model of compound interest; however, since this is a math course and not an economics course, we shall focus primarily on the practice of the math involved rather than the theory. Thus, we shall mostly deal with defining terminology without theory, and then seeing examples done.

Definition 1 (Compound Interest)**.** Suppose a principal P earns interest at the annual rate of r, and interest is compounded m times a year. Then, the amount F after t years is

$$F = P(1+i)^n = P\left(1 + \frac{r}{m}\right)^{mt}$$

where $n = mt$ is the number of time periods and $i = \frac{r}{m}$ is the interest per period.

Recall that the principal is what is deposited, the interest is how much is accrued after a certain period of time, and we say the interest is compounded if several time periods have passed for which interest is accrued on itself.

For convenience, we have the following cases that we may encounter listed here.

1. If interest is compounded yearly, $m = 1$.
2. If interest is compounded quarterly, $m = 4$.
3. If interest is compounded monthly, $m = 12$.
4. And so on.

Example 4. *Suppose $1000 is deposited into an account that yields 9% annually. Find the amount in the account at the end of the fifth year if the compounding is*

(a) *annually,*

(b) *quarterly,*

(c) *monthly.*

To solve this, we just need to apply the above definition with the correct values of m. Note that since we wish to find the amount at the end of the fifth year, 5 years will have passed and so $t = 5$. Next, since \$1000 is being invested, we have the principal is $P = 1000$. Lastly, since the account yields 9% annually, we have that $r = 0.09$. We note that we are to find F in the definition of compound interest.

(a) Since the compounding is done annually, we must take $m = 1$. Thus, we apply the definition and find

$$F = P\left(1 + \frac{r}{m}\right)^{mt} = 1000\left(1 + \frac{0.09}{1}\right)^{1\cdot 5} \sim 1538.62$$

is the amount in the account after 5 years have passed.

(b) Since the compounding is done quarterly, we must take $m = 4$. Thus, we apply the definition and find

$$F = P\left(1 + \frac{r}{m}\right)^{mt} = 1000\left(1 + \frac{0.09}{4}\right)^{4\cdot 5} \sim 1560.51$$

is the amount in the account after 5 years have passed.

(c) Since the compounding is done monthly, we must take $m = 12$. Thus, we apply the definition and find

$$F = P\left(1 + \frac{r}{m}\right)^{mt} = 1000\left(1 + \frac{0.09}{12}\right)^{12\cdot 5} \sim 1564.68$$

is the amount in the account after 5 years have passed.

△

This example shows how much one will accrue in interest given a specified amount deposited. What if we wanted to know how much one should invest to return a specified amount? This is essentially how we define present value.

Definition 2. Suppose an account earns an annual rate of r and compounds m times a year. Then, the amount P, is called the **present value**, needed currently in this account so that a future amount of F will be attained in t years is given by

$$P = \frac{F}{(1 + \frac{r}{m})^{mt}}.$$

Example 5. *Suppose you want %20K after 18 years, and you've found an account that earns 9% compounded quarterly. How much money should you set aside?*

Since we are compounding quarterly, we have $m = 4$. Since the account earns 9%, we have $r = 0.09$. Since we wish to determine how much money to invest in order to earn 20K in 18 years, we have $t = 18$. Lastly, since we want to earn 20K, we have $F = 20,000$. Thus, applying the definition of present value, we have

$$P = \frac{F}{(1 + \frac{r}{m})^{mt}} = \frac{20,000}{(1 + \frac{0.09}{4})^{4\cdot 18}} \sim 4029.69.$$

Thus, under the given conditions, to earn 20K after 18 years, we need to invest approximately 4029 initially. △

1.2 Continuous Compounding

Here we see what happens when we so called compound continuously. The question we are essentially asking is, What if we compound not discretely (e.g., monthly, quarterly, daily), but rather continuously (think infinitesimally)? It's a wonderful result that a constant (like π) pops up in this theory. The constant is e and is approximately equal to 3.

Definition 3 (Continuous compounding)**.** If a principal P earns interest at the annual rate of r, and interest is compounded continuously, then the amount F after t years is $F = P^{rt}$.

Example 6. *Suppose \$1000 is invested at an annual rate of 9% compounded continuously. How much is in the account after*

(a) *1 year.*

(b) *3 years.*

(a) Like the previous problems, this is just a matter of applying the above definition. We have $P = 1000$, $r = 0.09$, and $t = 1$. Thus,

$$F = Pe^{rt} = 1000e^{0.09\cdot 1} \sim 1094.17$$

(b) Here $P = 1000$, $r = 0$, and $t = 3$. Thus,

$$F = Pe^{rt} = 1000e^{0.09\cdot 3} \sim 1309.96.$$

△

The last thing we remark on is the present value for continuous compounding. Note that the process is almost identical to the discrete compounding above.

Definition 4 (Present value for continuous componding)**.** Suppose an account earns an annual rate of r, and compounds continuously. Then the amount P, called the present value, needed presently in this account so that a future amount of F will be attained in t years is given by

$$P = Fe^{-rt}.$$

Example 7. *Suppose you want %20K after 18 years, and you've found an account that earns 9% compounded continuously. How much money should you set aside?*

Since the account earns 9%, we have $r = 0.09$. Since we wish to determine how much money to invest in order to earn 20K in 18 years, we have $t = 18$. Lastly, since we want to earn 20K, we have $F = 20,000$. Thus, applying the definition of present value, we have

$$P = 20000e^{-0.09\cdot 18} \sim 3957.97$$

Thus, under the given conditions, to earn 20K after 18 years, we need to invest approximately 3957 initially. △

2 Logarithms

2.1 Some properties of logarithms

In this section we explore a function that essential "undoes" what exponentiation does.

Definition 5 (Logarithm)**.** Let $a > 0$, $a \neq 1$. If $x > 0$, then the logarithm base a of x, denoted $\log_a x$, is defined as the number y such that $x = a^y$; i.e.,

$$y = \log_a x \text{ if and only if } x = a^y.$$

It turns out that the most commonly used logarithm is the log based e, denoted ln. This logarithm is called the natural logarithm. The next most commonly used is log based 10, which is denoted by log. Before we see examples, let us remark on two properties of the logarithm first.

Theorem 2. *If $a > 0, a \neq 1$, then $a^{\log_a x} = x$ for all $x > 0$. Moreover, $\log_a a^x = x$ for all x.*

Example 8. *Compute the following*

(a) $10^{\log 4}$

(b) $24^{\log_{24} 15}$

(c) $\log_\pi \pi^5$

(d) $\log_3 3^{-25}$

(e) $\ln e = 1$.

To do the computations, we really just need to apply the previous theorem.

(a) $10^{\log_{10} 4} = 4$.

(b) $24^{\log_{24} 15} = 15$.

(c) $\log_\pi \pi^5 = 5$.

(d) $\log_3 3^{-25} = -25$.

△

We explore a couple more properties of logs.

Theorem 3. *For x, y, a sufficiently restricted,*

(a) $\log_a xy = \log_a x + \log_a y$

(b) $\log_a \frac{x}{y} = \log_a x - \log_a y$

(c) $\log_a x^c = c \log_a x$.

Example 9. *Compute the following*

(a) $e^{2 \ln 6}$

(b) $4^{3 \log_4 3}$

(c) $\log_{15} \frac{1}{15^3}$

(d) $\log_{7.2} \frac{1}{7.2^2}$.

△

Change of base

In this section we explore a specific property of logarithms that allows us to change from bases.

Theorem 4 (Change of Base)**.** *Let x, a, b be sufficiently restricted. Then*

$$\log_b x = (\log_b a)(\log_a x).$$

That is, we can write the logarithm of x based a as a logarithm of x based b, and vice verse. This is also sometimes written as

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

The importance of this is to provide you with a way of calculating a weird based logarithm with your calculator, as the next theorem shows.

Theorem 5. *When wanting to compute $\log_a x$ with $a \neq 10$, one may use*

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}$$

in their calculator to compute the logarithm.

2.2 Solving Logarithmic equations

In this section we explore how one would solve for "x" in an equation involving logarithms or exponents. We do this by example.

The first example will concern solving an exponential equation in full generality.

Example 10. *Solve for x in the equation*

$$3(4^x) = 5.3,$$

and express your answer rounded to four decimal places.

The point here is that there is no way of solving this equation like we did before by changing the bases. We have to use logarithms here. It turns out that there are several ways to do this problem. We show one way here.

First, isolate the 4^x term:

$$4^x = \frac{5.3}{3}.$$

Second, take log based 4 on *both* sides:

$$\log_4 4^x = \log \frac{5.3}{3}.$$

Third, we recall that $\log_4 4^x = x$, giving

$$x = \log_4 \frac{5.3}{3}.$$

Lastly, we compute $\log_4 \frac{5.3}{3}$ with a calculator and round to four decimal places

$$\log_4 \frac{5.3}{3} \sim 0.4105.$$

Thus,

$$x \sim 0.4105$$

is the desired answer. △

Example 11. *Solve for x in*

$$\log_3 x = 4.$$

The trick here is to just exponentiate by 3:

$$3^{\log_3 x} = 3^4,$$

noting that $3^{\log_3 x} = x$, and so $x = 3^4$. △

Example 12. *Solve for x in the expression*

$$\log_3(x) = \frac{1}{2}\log_3 36 + \log_3 2 - \log_3 4.$$

The idea here is to firstly combine the right hand side into a single logarithm using the rules listed before. We first treat $\frac{1}{2}\log_3 36$:

$$\frac{1}{2}\log_3 36 = \log_3 36^{1/2} = \log_3 6,$$

thus,

$$\log_3(x) = \log_3 6 + \log_3 2 - \log_3 4.$$

Next, we combine $\log_3 6 + \log_3 2$:

$$\log_3 6 + \log_3 2 = \log_3(6 \cdot 2) = \log_3 12,$$

thus

$$\log_3(x) = \log_3 12 - \log_3 4.$$

Next, we must combine $\log_3 12 - \log_3 4$:

$$\log_3 12 - \log_3 4 = \log_3 \frac{12}{4} = \log_3 3 = 1,$$

thus,

$$\log_3(x) = 1.$$

Lastly, we exponentiate to get rid of the logarithm,

$$x = 3^{\log_3 x} = 3^1 = 3,$$

and so $x = 3$ is the desired solution. △

Example 13. *Solve for x in the expression*

$$\log_3(3x+1) + \log_3(3x-1) = 2\log_3(2x).$$

The goal here is to express each side as a single logarithm, then exponentiate, and the solve the new equation for x. I note this now because this will be crucial: in what follows, we are at best finding potential solutions–I will say more on this at the end. So, we first combine $\log_3(3x+1)$ and $\log_3(3x-1)$:

$$\log_3(3x+1) + \log_3(3x-1) = \log_3[(3x+1)(3x-1)] = \log_3(9x^2-1).$$

Thus, we have

$$\log_3(9x^2-1) = 2\log_3(2x).$$

Next, we bring the 2 into $\log_3(2x)$:

$$2\log_3(2x) = \log_3(2x)^2 = \log_3 4x^2,$$

and so

$$\log_3(9x^2-1) = \log_3 4x^2.$$

We now exponentiate by 3:

$$9x^2 - 1 = 3^{\log_3(9x^2-1)} = 3^{\log_3 4x^2} = 4x^2.$$

We are thus left to solve,

$$5x^2 - 1 = 0.$$

Thus, $x = \pm\sqrt{\frac{1}{5}}$ are potential solutions. We now need to make sure that any of the logarithms in the equation are defined for either of these values, which can be done using a calculator. We need only check that $\log_3(9x^2-1)$ and $\log_{2x}$ are defined for these values of x. We find that

$$9(\pm\frac{1}{\sqrt{5}})^2 - 1 = \frac{9}{5} - 1 > 0$$
$$-2\frac{1}{\sqrt{5}} < 0$$
$$2\frac{1}{\sqrt{5}} > 0.$$

This shows that only $\frac{1}{\sqrt{5}}$ is a solution. △

Example 14. *Solve for x in*

$$\log_4(2x+1) = 2 + \log_4 2$$

Here, we will do similarly to above, but we will need to handle the term 2. First, move all logarithms to one side:

$$\log_4(2x+1) - \log_4(2) = 2.$$

Next, combine logarithms:

$$\log_4(2x+1) - \log_4(2) = \log_4 \frac{2x+1}{2} = 2.$$

Lastly, exponentiate by 4 to eliminate the logarithms:

$$\frac{2x+1}{2} = 4^{\log_4 \frac{2x+1}{2}} = 4^2 = 16.$$

Thus, we need only solve

$$\frac{2x+1}{2} = 16.$$

So, $x = \frac{35}{2}$. △