1 Limits and Continuity

1.1 Finding limits from a graph

Cf. lecture: lack of limits as f approach $\pm\infty$, when one sided limits disagree, when they agree despite function not being defined there, etc.

Definition 1 (One sided limits)**.** Let f be a function and a a real number. Then, the right sided limit of $f(x)$ as x approaches a, denoted $\lim_{x\to a^+} f(x)$, is the value $f(x)$ approaches as $x \to a$ from the right of a. Similarly, the left sided limit, denoted $\lim_{x\to a^-} f(x)$, is the value $f(x)$ approaches as $x \to a$ from the left of a.

Definition 2 (Limit)**.** If

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = A,$$

then we say A is the limit of $f(x)$ as x approaches A, and denote this by

$$\lim_{x\to a} f(x) = A.$$

1.2 Finding limits of a polynomial

It turns out, as we will discuss, that polynomial functions are always continuous. And thus, by consequence of being continuous (as we shall note), finding limits of polynomial function turn out to be really easy. Indeed, if f is a polynomial function, then

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a} f(x) = f(a)$$

for every real number a.

Example 1. *Compute the limits*

$$\lim_{x\to 4}(x^3 + 2x - 1).$$

As mentioned above, it turns out that, since our function of interest is a polynomial, we can find this limit by simply inputing 4 for x:

$$\lim_{x\to 4}(x^3 + 2x - 1) = 4^3 + 2(4) - 1 = 71.$$

△

For emphasis, we restate this property of polynomials as a theorem.

Theorem 1 (Limits of polynomials)**.** *Let $p(x)$ be a polynomial function. Then,*

$$\lim_{x\to a} p(x) = p(a)$$

for all real numbers a.

1.3 Finding limits of rational functions

In the previous section, we dealt with taking limits of very nice functions. Here, we will attempt to take limits of rational functions (note that polynomials are technically rational functions, but for this section, we take rational functions to mean rational functions that have non-constant polynomials in the denominators). It turns out taking limits of rational functions are usual as straightforward as taking limits of polynomials, as the next theorem states.

Theorem 2. *Let $r(x)$ be a rational function with $r(a)$ defined. Then*

$$\lim_{x\to a} r(x) = r(a).$$

We note that even though this theorem is true, since this class focuses on problem solving rather than material, the work necessary for solving such limit problems will be seen in the next example. But, before we can see this, we need a nice property of limits.

Theorem 3. *Let f and g be functions, suppose* $\lim_{x\to a} f(x) = A$ *and* $\lim_{x\to a} g(x) = B \neq 0$. *Then,*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}.$$

Example 2. *Compute the following limit*

$$\lim_{x\to 3} \frac{x^2 + x}{x + 3}.$$

To compute this limit, we shall apply the previous theorem. So, let $f(x) = x^2 + x$ and $g(x) = x + 3$. Then, we wish to compute

$$\lim_{x\to 3} \frac{f(x)}{g(x)}.$$

We find that, since f and g are polynomials, that

$$\lim_{x\to 3} f(x) = f(3) = 3^2 + 3 = 12,$$

and

$$\lim_{x\to 3} g(x) = g(3) = 3 + 3 = 6 \neq 0.$$

Thus, we may apply the previous theorem and obtain,

$$\lim_{x\to 3} \frac{x^2 + x}{x + 3} = \lim_{x\to 3} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 3} f(x)}{\lim_{x\to 3} g(x)} = \frac{12}{6} = 2.$$

Note that this sort of computation (though one can write it a bit more concise), is the sort of work looked for. △

The next question to ask is, If $r(x)$ is a rational function and, say, $r(a)$ is undefined, how do we compute $\lim_{x\to a} r(x)$? It turns out that this limit may or may not exist; i.e., whether or not the limit exists is entirely independent of whether or not $r(a)$ exists. To answer this question, we show by example the so-called *method of cancellation.*

Example 3. *Compute the limit*

$$\lim_{x\to 8} \frac{x^2 + x - 72}{x - 8}.$$

Letting $r(x) = \frac{x^2+x-72}{x-8}$, we note that $r(8)$ is not defined. However, it turns out that the limit does exist because of the following reasoning. We find that

$$r(x) = \frac{x^2 + x - 72}{x - 8} = \frac{(x - 8)(x + 9)}{x - 8} = x + 9 \qquad \text{for } x \neq 8.$$

Note that the above equality holds for $x \neq 8$. In any case, this shows that

$$\lim_{x\to 8} r(x) = \lim_{x\to 8} x + 9 = 8 + 9 = 17.$$

(If you want more justification, draw the graph of the function

$$f(x) = \begin{cases} x + 9 & x < 8 \\ x + 9 & x > 8 \end{cases}$$

and analyze the limit graphically.)

To recap, we factored as much as possible and canceled factors where possible. This cancellation is precisely why the limit existed. △

Even though this previous example showed that the limit existed, this need not be the case. In the next example we see that it may be the case for a limit to not exist.

Example 4. *Determine whether or not the following limit exists*

$$\lim_{x\to 0} \frac{1}{x}.$$

The idea here is to show that $\lim_{x\to 0^+} \frac{1}{x}$ and $\lim_{x\to 0^-} \frac{1}{x}$ disagree, and therefore the limit does not exist. That is, we first compute the left and right sided limits.

We find that as $x \to 0^+$, x is always positive, and therefore, $\frac{1}{x} > 0$. It follows that, since x is decreasing, $\frac{1}{x}$ is increasing, and so

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

Similarly, as $x \to 0^-$, x is always negative, and therefore, $\frac{1}{x} < 0$. It follows that, since x is decreasing, $\frac{1}{x}$ is increasing,a nd so

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Therefore,

$$\lim_{x\to 0} \frac{1}{x}$$

does not exist as the one sided limits do not agree. △

Example 5. *Determine whether or not the following limit exists*

$$\lim_{x\to 4} \frac{x^2+2x+1}{x-4}.$$

First, let $r(x) = \frac{x^2+2x+1}{x-4}$ and note $r(4)$ does not exist. Thus, in order to determine whether or not the limit exists, we need some analysis. Next, as above, to do this problem, we first factor as much as possible and see if any cancellation occurs. We find that

$$r(x) = \frac{x^2+2x+1}{x-4} = \frac{(x+1)^2}{x-4},$$

and so, in particular, the $x-4$ factor does not cancel it.

Thus, what is left is to analyze what happens as $x \to 4^-$ and $x \to 4^+$. Now, as $x \to 4^+$, $x - 4 > 0$ as $x > 4$. It follows that since $(x+1)^2 \geq 0$ that $\frac{(x+1)^2}{x-4} > 0$. Therefore, as $x \to 4^+$, $r(x)$ is approaching $+\infty$; i.e.,

$$\lim_{x\to 4^+} \frac{(x+1)^2}{x-4} = +\infty.$$

Similarly, as $x \to 4^-$, $\frac{(x+1)^2}{x-4} < 0$, and so

$$\lim_{x\to 4^-} \frac{(x+1)^2}{x-4} = -\infty.$$

Thus, since the one sided limits do not agree, the limit

$$\lim_{x\to 4} r(x)$$

does not exist. △

Example 6. *Compute the following limit*

$$\lim_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4}.$$

In this case, we will determine the limit does in fact exist by using cancellation. We first factor:

$$\frac{x^2+5x+4}{x^2+3x-4} = \frac{(x+4)(x+1)}{(x+4)(x-1)} = \frac{(x+1)}{(x-1)} \quad \text{for } x \neq -4.$$

Thus, we find that

$$\lim_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4} = \lim_{x\to -4} \frac{x+1}{x-1} = \frac{\lim_{x\to -4}(x+1)}{\lim_{x\to -4}(x-1)} = \frac{-4+1}{-4-1} = \frac{3}{5}.$$

△

1.4 Finding limits of piecewise functions

In this section, we explore the idea of computing limits of piecewise functions. In reality, there is nothing new in this section: we basically compute one sided limits of the case functions that make up the piecewise function, and then determine whether or not they agree. If the one sided limits agree, the piecewise function has a limit, otherwise, it does not. We shall do this section by example. Let's see an example of this.

Example 7. *Determine the following limits for the function defined below (If an answer does not exist, write DNE):*

$$f(x) = \begin{cases} x+1 & x < 5 \\ \sqrt{x+31} & x > 5 \end{cases}.$$

(a) $\lim_{x\to 5^+} f(x)$

(b) $\lim_{x\to 5^-} f(x)$

(c) $\lim_{x\to 5} f(x)$.

The trick here, as usual, is to determine

$$\lim_{x\to 5^+} f(x)$$

and

$$\lim_{x\to 5^-} f(x),$$

and determine if they are equal. But, we must do this by a sort of case-by-case analysis. For $x > 5$ with x close enough to 5, $f(x) = \sqrt{x+31}$ by definition. Moreover, if $x < 5$ and x is close enough to 5, then $f(x) = x + 1$ by definition. It follows that

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} x+1$$

and

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \sqrt{x+31}.$$

We may thus answer the above questions.

(a) Since the function $g(x) = \sqrt{x+31}$ is continuous at $x = 5$, we find

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5} g(x) = g(5) = \sqrt{5+31} = 6.$$

(b) Since the function $h(x) = x + 1$ is continuous at $x = 5$, we find

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5} h(x) = h(5) = 5 + 1 = 6.$$

(c) Now, since the left and right sided limits agree, we find that

$$\lim_{x \to 5} f(x) = 6.$$

We remark on the fact that $f(5)$ isn't defined despite of the limit being defined. △

Example 8. *Determine the following limits for the function defined below (If an answer does not exist, write DNE):*

$$f(x) = \begin{cases} 1 - x^3 & x < 5 \\ x + 5 & x \geq 5 \end{cases}.$$

(a) $\lim_{x \to 5^+} f(x)$

(b) $\lim_{x \to 5^-} f(x)$

(c) $\lim_{x \to 5} f(x)$.

Like before, we analyze the limits case by case. For $x \to 5^+$, $f(x) = x + 5$ since $x \geq 5$. Similarly, for $x \to 5^-$, $f(x) = 1 - x^3$ since $x < 5$. We can then answer the questions.

(a) By the work above, we conclude

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (x + 5) = 5 + 5 = 10.$$

(b) Also, we conclude that

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (1 - x^3) = 1 - 5^3 = -124.$$

(c) Thus, since the right and left sided limits do not agree, we conclude that the limit $\lim_{x \to 5} f(x)$ DNE.

△

1.5 Finding where a function is continuous or discontinuous

Definition 3 (Continuity)**.** If

$$\lim_{x \to a} f(x) = f(a),$$

then f is said to be continuous at $x = a$.

As mentioned above, a function f is continuous at point a precisely when the limit of $f(x)$ as $x \to a$ equals $f(a)$. It turns out that it is quite simple finding where polynomials, rational, and piecewise functions (so long as they are nice enough) are continuous. In fact, finding where a function is continuous (for us) amounts to computing limits where needed, or by use of inspection. Before we see an example, let's state a theorem.

Theorem 4.

(i) A polynomial function is continuous everywhere.

(ii) A rational function is continuous everywhere it is defined.

(iii) A piecewise function f is continuous everywhere its cases are continuous and where $f(a) = \lim_{x \to a} f(x)$, where a is an endpoint of any domain of its cases.

(iv) A radical is continuous where it is defined.

Let's see some examples.

Example 9. The function $f(x) = \frac{1}{x}$ is continuous everywhere except at $x = 0$. That is, if we were to write where f is continuous in interval notation, we would say f is continuous on $(-\infty, 0) \cup (0, +\infty)$. △

Example 10. *Determine where* $f(x) = \frac{x+1}{x^2-1}$ *is continuous.*

To answer this question, we first factor:

$$\frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)}.$$

Thus we see that $f(1)$ and $f(-1)$ are not defined and so we conclude that f is continuous everywhere except for $x = -1, 1$; i.e., f is continuous on $(-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$. △

Definition 4 (Discontinuous)**.** A function is discontinuous wherever it is undefined or not continuous.

Theorem 5.

(i) A polynomial is never discontinuous.

(ii) A rational function is discontinuous only where it is undefined.

(iii) A piecewise function is discontinuous where its cases are discontinuous, and where it is undefined.

Example 11. The function $f(x) = 1/x$ is discontinuous at $x = 0$. △

Example 12. *Determine where* $f(x) = \frac{8x^2+2x+1}{x^3+5x^2+6x}$ *is discontinuous.*

In finding where a function is continuous, we first determined where the function has "bad" points. Here, we are done after finding these "bad" points. Thus, as f is a rational function, we need only check where f is undefined. We first factor the denominator (note we care not about the numerator!)

$$f(x) = \frac{8x^2 + 2x + 1}{x(x^2 + 5x + 6)} = \frac{8x^2 + 2x + 1}{x(x+3)(x+2)},$$

and so f is discontinuous only at $x = 0, -3, -2$. (Note, then, that f is continuous on $(-\infty, -3) \cup (-3, -2) \cup (-2, 0) \cup (0, \infty)$.) △

Example 13 (Graphical example)**.** *Refer to class.* △

Example 14. *Determine where the piecewise function*

$$f(x) = \begin{cases} \sqrt{x+1} & x < 0 \\ x + 1 & 0 \leq x < 5 \\ x^2 & 5 \leq x < 7 \\ x^2 & x > 7 \end{cases}$$

is continuous and where it is discontinuous.

The point of this problem is to enumerate most of the nuances with finding where a piecewise function is continuous or discontinuous. The method of solving this problem is rather algorithmic. We first check where the piecewise is undefined (as function are discontinuous wherever they are undefined). In the first case (i.e., $x < 0$), we note that $\sqrt{x+1}$ is undefined for $x + 1 < 0$; i.e., when $x < -1$. Thus, at the very least, f is discontinuous on $(-\infty, -1)$. Next, we see that f is in fact defined everywhere else except at $x = 7$, and therefore, we find that f is discontinuous on at least $(-\infty, -1)$ and at $x = 7$. Thus, since each of the cases in f are defined everywhere else, we need only check the limits at the endpoints of the intervals on which the cases are defined.

The endpoints of where the cases are defined are the values $x = 0, 5, 7$. Now, since the cases are nice enough functions, we can actually find their respective limits as $x \to 0, 5, 7$ from their respective sides. For example, considering when $x < 0$, we find that, since $\sqrt{x+1}$ is continuous at 0,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{x+1} = \sqrt{0+1} = 1.$$

Next, for $0 \leq x < 5$, we find that, since $x + 1$ is continuous at 0,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x + 1 = 0 + 1 = 1.$$

Lastly, we find that $f(0) = 0 + 1 = 1$. Therefore, f is continuous at $x = 0$.

The next endpoint we need to consider is $x = 5$. We thus compute the limit

$$\lim_{x \to 5^- f(x)} = \lim_{x \to 5^-} x + 1 = 5 + 1 = 6.$$

But, $f(5) = 5^2 = 25$. Thus, it cannot possibly be the case that $\lim_{x \to 5} f(x) = f(5)$, and so we conclude that f is discontinuous $x = 5$.

Lastly, we consider the endpoint $x = 7$. However, we already noted that $f(7)$ isn't even defined and therefore we are done; i.e., f is necessarily discontinuous at $x = 7$. $\triangle$