1 Rates of Change and Tangent Lines

In this section we shall formulate rate of change mathematically and consider a geometric object called the tangent line of a function. It may become surprising that something as tangible as rate of change and something purely mathematical and geometric should be considered in the same section. It turns out that these two subjects are very related (and in fact, in a sense, equivalent notions) and can be considered as the back bone of calculus.

1.1 Average Rate of Change

It is pretty useful to know and understand the behavior of how a function changes (and how quickly) with respect to some dependent variable. We can usually get a pretty decently approximated answer to the question, How "quickly" is the quantity $f(x)$ changing with respect to x? Recall that that the slop of a linear function told you how much "rise" the function rose when a certain amount of "run" was ran. In other words, the quantity $\frac{\text{rise}}{\text{run}} = \frac{f(y)-f(x)}{y-x}$ of a linear function f measured how much the output would change as we changed the input by $y-x$. That is, the "rise over run" of the linear function measures the rate of change of the function.

Well, given a function, we can approximate the function by a linear function and use this idea of "rise over run" to approximate the rate of change of the function.

Definition 1.1.1. The **average rate of change** of $y = f(x)$ with respect to x as x changes from a to b (a.k.a., over $[a, b]$, over (a, b)) is the quantity

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

We note that the quantity $\frac{f(b)-f(a)}{b-a}$ is in fact the slope of the line that passes through the two points $(a, f(a))$ and $(b, f(b))$. This line has the name **secant line**; i.e., the average rate of change of a function from a to b is given by the slope of the corresponding secant line. To add onto this, we note that a negative average rate of change indicates an overall average decrease in the function, and a positive rate of change indicates an overall average increase in the function.

Luckily, when finding the average rate of change of a function, you are really just inputing numbers into some formula.

Example 1.1.2 (Average rate of change of a function)**.** Let $f(x) = \frac{1}{x+2}$. We shall find the average rate of change as x changes from 2 to 5. To do so, we simply input the appropriate values into the appropriate formula,

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(5) - f(2)}{5 - 2} = \frac{\frac{1}{5+2} - \frac{1}{2+2}}{5 - 2} = -\frac{1}{28}.$$

Thus, as x changes from 2 to 5, the output values of f will change over this interval by $-\frac{1}{28}$, on average. We should expect this number to be negative since as x increase from 2 to 5, $f(x)$ decreases from $\frac{1}{4}$ to $\frac{1}{7}$.

Example 1.1.3. *Calculate the average rate of change of the given function over the interval* $[1, 3]$ *and specify the units of measurement*

x(seconds)	1	2	3	4	5
$f(x)$(meters)	2	4	6	3	8

The only difference here from the previous example is that instead of given a function via an equation, we are given a function via a table. Thus, to find the average rate of change, we still only need to apply the definition of average rate of change; i.e., we simply need to extract from this problem the appropriate $a, b, f(a)$, and $f(b)$. But, as before, a and b come from the specified interval and so we have $a = 1$ and

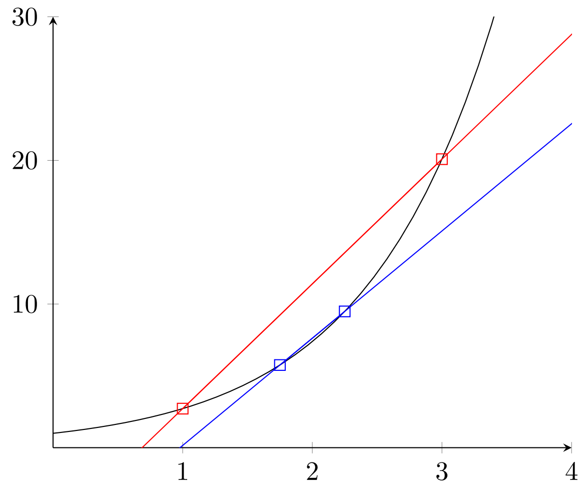

Figure 1: Two secant lines of $f(x) = e^x$

$b = 3$; i.e., x is changing from 1 to 3. Next, we read the table to find what $f(a)$ and $f(b)$ are: we find that $f(a) = f(1) = 2$ and $f(b) = f(3) = 6$. Thus, we are left with inputing this data into the formula:

$$\frac{\Delta y}{\Delta x} = \frac{f(3) - f(1)}{3 - 1} = \frac{6 - 2}{3 - 1} = 2.$$

Therefore, we have found that the average rate of change is 2. But what does this mean in terms of the units of measurement? Well, x was given as a time variable in seconds, and f was given in meters. Therefore, the average rate of change measured the average rate of change in meters per second; i.e., the unites of measurement is m/s. To put it another way, we have found, on average, the speed of something.

1.2 Instantaneous Rate of Change

In the definition of average rate of change, if we take a and b ever so close, we can get a pretty good approximation of the function. To see this, consider the graph and secants in Figure 1. The graph is of the function $f(x) = e^x$, the two red squares are at the points $(1, e)$ and $(3, e^3)$, which give the secant line drawn as the red line through these points. The slope of the red secant line is the average rate of change of f over $(1, 3)$, and we see that this line approximates f very terribly. However, the blue secant line given by the points $(1.75, e^{1.75})$ and $(2.25, e^{2.25})$, drawn as the blue squares, seems to approximate f quite nicely.

Now imagine taking a and b even closer (e.g., 1.999 and 2.0001), we see that the secant lines are closing in on the point $(2, e^2)$. That is, imagine taking the limit of secant lines in this fashion. One might argue that, in the limit, we are approximating the rate of change of f at $x = 2$, and this is what we call the instantaneous rate of change. To put it in a formal definition,

Definition 1.2.1. Let c be in the domain of a function f. Then the **instantaneous rate of change** of f at c is given by the limit

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h},$$

provided this limit exists.

Two things to note about this definition. It is possible for a function to not have an instantaneous rate of change if the given limit does not exist (more on this later). Second, this limit does in fact come from the description above. To see this, consider letting $h = b - a$. Then,

$$\frac{f(c + h) - f(c)}{h} = \frac{f(c + b - a) - f(c)}{c + b - a - c},$$

which is the slope of the secant line through the points $(a+b-a, f(c+b-a)$ and $(c, f(c))$, so that taking $b-a \to 0$ (i.e., taking a and b closer and closer and so also taking $h \to 0$), and so limiting the secant lines as described above, we get the instantaneous rate of change in the limit.

Let's see some examples of computing the instantaneous rate of change.

Example 1.2.2 (Instantaneous rate of change of x)**.** Let's start out with an easy function. Let $f(x) = x$ and suppose we wish to find the instantaneous rate of change of f at $x = 2$ (note that we are taking $c = 2$ in the definition). First, let's do some simplifying,

$$\frac{f(2+h)-f(2)}{h} = \frac{2+h-2}{h} = \frac{h}{h} = 1.$$

Then,

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} 1 = 1.$$

Thus, the instantaneous rate of change of f at $x = 2$ is 1.

Before we go on, note that the final limit does not didn't depend on c being equal to 2. In fact, the instantaneous rate of change was completely independent of which point we chose and so the instantaneous rate of change at *any* c is 1.

Example 1.2.3 (Instantaneous rate of change of x^2)**.** Let's move up in difficulty. Let $g(x) = x^2$ and find the instantaneous rate of change of g at $x = 4$. We again do some simplification first,

$$\frac{g(4+h)-g(4)}{h} = \frac{(4+h)^2-4^2}{h} = \frac{16+8h+h^2-16}{h} = 8+h.$$

Then,

$$\lim_{h\to 0} \frac{g(4+h)-g(4)}{h} = \lim_{h\to 0}(8+h) = 8.$$

Thus, the instantaneous rate of change of g at $x = 4$ is 8. This time, however, the instantaneous rate of change did depending on c. To see this, let's do find the instantaneous rate of change of g more generally.

$$\frac{g(c+h)-g(c)}{h} = \frac{(c+h)^2-c^2}{h} = \frac{c^2+2ch+h^2-c^2}{h} = 2c+h.$$

Thus, the instantaneous rate of change of g at c is

$$\lim_{h\to 0} \frac{g(c+h)-g(c)}{h} = \lim_{h\to 0}(2c+h) = 2c.$$

Example 1.2.4 (Instantaneous rate of change of $\frac{1}{x}$)**.** Let's continue in increasing difficulty. Let $f(x) = \frac{1}{x}$. Now, instead of finding the instantaneous rate of change of f at some value for c, let us keep c arbitrary (though $c \neq 0$ since $f(0)$ is undefined) as done at the end of Example 1.2.3. First, let's do the simplification,

$$\frac{f(c+h)-f(c)}{h} = \frac{\frac{1}{c+h}-\frac{1}{c}}{h} = \left(\frac{1}{c+h}-\frac{1}{c}\right)\frac{1}{h} = \left(\frac{c-c-h}{c(c+h)}\right)\frac{1}{h} = -\frac{h}{c(c+h)}\frac{1}{h} = -\frac{1}{c(c+h)}.$$

Thus, the instantaneous rate of change of f at any c such that $c \neq 0$ is given by

$$\lim_{h\to 0} \frac{1}{c(c+h)} = -\frac{1}{c^2}.$$

Example 1.2.5. *Find the instantaneous rate of change of* $f(x) = x^2 + 2x + 1$ *at* $x = 4$*.*

To do this problem, we directly apply the definition of instantaneous rate of change. We find

$$\lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{(4+h)^2+2(4+h)+1-4^2-8-1}{h}$$

$$= \lim_{h\to 0} \frac{4^2 + 8h + h^2 + 24 + 2h + 1 - 4^2 - 8 - 1}{h}$$

$$= \lim_{h\to 0} \frac{8h + h^2 + 2h}{h} = \lim_{h\to 0}(10 + h) = 10.$$

Thus the instantaneous rate of change at $x = 4$ is 10.

Now, if you have seen calculus before, this might be familiar in that we are really just computing derivatives here by a limit. In any case, across the country it is insisted (for some unidentifiable mysterious reason–it's a common problem students fail at doing) to ask on exams for students to compute limits like these, and so it is important (for most students) to learn how to compute these limits and their variations for various functions (e.g., $\frac{1}{x^2}$, $\frac{1}{x+3}$, $(x + 2)$, etc).

1.3 Tangent Lines

We now move on to the geometric side of this discussion, though we have already seen some geometry in the secant lines! In the same way we gave the secant line a name, we to give name for the line obtained by taking the limit of secants lines as done earlier. Also, in the same way the average rate of change gave the slope of secant lines, the instantaneous rate of change gives the slope of this limiting line. The line so obtained will be called a tangent line, and it is the line whose slope is given by the aforementioned limit of secant slopes, and who passes through the point of the graph the secant lines are tending to. To put it more concretely and formal, let's write it as a definition.

Definition 1.3.1. Let c be in the domain of a function f. Then the **tangent line** of f at $(c, f(c))$ has slope given by the limit

$$m_{tan} = \lim_{h\to 0} \frac{f(c + h) - f(c)}{h},$$

provided this limit exists, and it thus given by the equation

$$y_{tan} = m_{tan}(x - c) + f(c).$$

We note that the structure of this definition is also identical to that of the definition given by instantaneous rate of change. Therefore, if asked to find the tangent line of a function f through $(c, f(c))$, we really only need to find the instantaneous rate of change of f at c and input our data into the equation for y_{tan}.

Example 1.3.2 (Tangent line of x^2)**.** Let's find the tangent line of the function $f(x) = x^2$ through the point $(4, f(4)) = (4, 16)$. In Example 1.2.3, we found that the instantaneous rate of change of f at $x = 4$ was 8, and therefore, the slope of tangent line of interest is given by

$$m_{tan} = \lim_{h\to 0} \frac{f(4 + h) - f(4)}{h} = 8.$$

Hence, by $c = 4$, $f(c) = 16$, and $m_{tan} = 8$, we find that the tangent line of interest is given by the equation

$$y = 8(x - 4) + 16 = 8x - 16.$$

What this looks like graphically is given in Figure 2. Note that the tangent line (given by the red line) just touches the graph at the single point $(4, 16)$ (given by the blue square)–this is always true if the function is nice enough and if we only look locally enough.

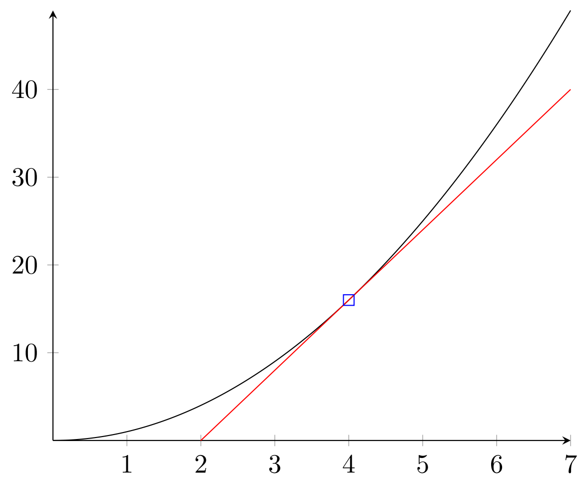

Figure 2: Tangent line of $f(x) = x^2$