1 Functions

1.1 Some Set Theory

What exactly *is* set theory will not worry us here. We care only about the notational usage of set theory (whatever that is) and we shall list this usage here. So let's see our first definition!

Definition 1.1.1. By a **set** A, we mean any (abstract) collection of (abstract) objects.

For example, the "set" of all real numbers contains (and you might guess) all real numbers. The set of all integers only contains integers and nothing else.

We will focus entirely on intervals and unions of intervals, so let us define some terminology.

Definition 1.1.2. Let a and b be any reals number or $\pm\infty$. We define the following types of intervals

(i) Open interval: (a, b) represents the set of all x such that $a < x < b$,

(ii) Half open intervals: $[a, b)$ or $(a, b]$ represent the set of all x such that $a \leq x < b$ or $a < x \leq b$, respectively, and

(iii) Closed intervals: $[a, b]$ represents the set of all x such that $a \leq x \leq b$.

Example 1.1.3.

1. The open interval $(0, 1)$ consists of all real numbers between 0 and 1; e.g., 1.5.
2. The open interval $(-\infty, 3)$ represents all real numbers less than 3.
3. The half open interval $[-8, \infty)$ represents all real numbers greater than or equal to -8.

We explicitly write all the possible cases of intervals that we will consider.

Interval Notation	Inequality Notation	Explanation
$a < x < b$	(a, b)	All x greater than a and less than b
$a < x \leq b$	$(a, b]$	All x greater than a and less than or equal to b
$a < x < \infty$ or $a < x$	(a, ∞)	All x greater than a
$a \leq x < b$	$[a, b)$	All x greater than or equal to a and less than b
$a \leq x \leq b$	$[a, b]$	All x greater than or equal to a and less than or equal to b
$a \leq x < \infty$ or $a \leq x$	$[a, \infty)$	All x greater than or equal to a
$-\infty < x < b$ or $x < b$	$(-\infty, b)$	All x less than b
$-\infty < x \leq b$ or $x \leq b$	$(-\infty, b]$	All x less than or equal to b
$-\infty < x < \infty$	$(-\infty, \infty)$	All x

We would like for notational convenience to have a notation defined for combining two or more sets.

Definition 1.1.4. Let A and B be two sets. Then we write $A \cup B$ to mean the set that contains elements of both A *and* B.

Example 1.1.5.

1. $(0, 1) \cup (.5, 4) = (0, 4)$.
2. $(-3, 4) \cup [4, 5) = (-3, 5)$,
3. $(-\infty, 1) \cup (3, 4)$ cannot be simplified,
4. $(2, 3) \cup (100, 300)$ cannot be simplified.

To see this equalities hold, you can either draw a number line or analyze the elements explicitly.

Example 1.1.6. The set of all real numbers can be represented by $(-\infty, \infty)$ or all x such that $-\infty < x < \infty$.

Note that understanding both of notations and unioning will be very important for later material.

1.2 Functions

We are now ready to handle functions more mathematically and in a way we will need to for the rest of the material. Let's finally define *what* a function *is*.

Definition 1.2.1. Let f be a rule that assigns to each element of X one and only one element of Y. Then f is called a **function**, X is called the **domain**, and Y is called the **codomain**. The set of outputs is called the **range**.

Remark. The range is also characterized by being the collection of y for which $f(x) = y$ for some x in X.

Example 1.2.2.

1. Let $f(x) = x^2$ be defined on all real numbers. Then the domain of f is $(-\infty, \infty)$ and its range is $[0, \infty)$

2. Let $g(x) = x^3$ be defined on all real numbers. Then its domain is $(-\infty, \infty)$ and its ranges is $(-\infty, \infty)$.

1.3 A Convention

Before we go on, you might have noticed that explicitly writing the domain and codomain of a function is tedious, which is why we employ the following convention from now on,

Convention. Suppose we are given a function f without explicitly mentioning its domain nor codomain. We shall then always assume that the domain of f is the largest possible set for which f makes sense and that its codomain is $(-\infty, \infty)$, unless otherwise specified. Note that this says nothing about the range of f. □

What this means is that if were are given a function (by a graph or by an equation), we are to assume the domain of the function is the largest set that the function makes sense on.

Example 1.3.1. Suppose we are given the function f defined by the rule $f(x) = \frac{1}{x}$. Then, we *must* assume that the domain of f is $(-\infty, 0) \cup (0, \infty)$ since we cannot divide by zero. That is, this set is the largest possible set for which f makes sense.

Example 1.3.2. Suppose we are given the function g defined by the rule $g(x) = \sqrt{x}$. Then, we *must* assume that the domain of g is the set of x such that $0 \leq x$, since the square root of a negative number is not allowed. That is, this set is the largest possible set for which g makes sense.

Let's see some three more difficult examples.

Example 1.3.3. Let $f(x) = \frac{1+x}{\sqrt{x+1}}$. We wish to find the domain of this function and so we wish to find where the function is undefined. We note that the $1 + x$ in the numerator causes us no issue, but the $\sqrt{x+1}$ in the denominator potentially does. Firstly, the square root of a negative number is not defined and so we must firstly have $x + 1 \geq 0$. Therefore, at the very least, $x \geq -1$. Moreover, we cannot divide by zero and so we must have $\sqrt{x+1} \neq 0$, and so, since $x + 1 = 0$ at $x = -1$, we must have $x \neq -1$. Using these two results, we may conclude $x > -1$ or $(-1, \infty)$ is the domain of f.

Example 1.3.4. Let $f(x) = \frac{x+1}{(x^2+2x+1)}$. We wish to find the domain of this function and so we wish to find where the function is undefined. As above, $x + 1$ in the numerator causes no issue, but the $x^2 + 2x + 1$ in the denominator potentially does. What might occur? Well, if $x^2 + 2x + 1 = 0$, we would be dividing by zero, and therefore, we need to determine where $x^2 + 2x + 1$ does not equal to zero. So we factor, $x^2 + 2x + 1 = (x + 1)^2$; i.e., $x^2 + 2x + 1 \neq 0$ so long as $x \neq -1$. Therefore, the domain of f is $(-\infty, -1) \cup (-1, \infty)$.

Example 1.3.5. Let $g(x) = \frac{x+2}{x^2+3x+2}$. As above, we only care where the denominator's quadratic does not equal zero. Thus, we factor, $x^2 + 3x + 2 = (x + 1)(x + 2)$. But then, $g(x) = \frac{x+2}{(x+1)(x+2)} = \frac{1}{x+1}$, right?

Well, not quite! It turns out that g and the function $h = \frac{1}{x+1}$ are not quite the same. Consider what happens when we input -2 for x in g:

$$g(-2) = \frac{-2+2}{(-2)^2 - 6 + 2} = \frac{0}{0},$$

which is not defined! This may seem nit picky, and it is, but, at least by convention, we assume g and h are not the same function here. Therefore, even though $h(-2)$ makes sense, $g(-2)$ does not. We conclude that -2 and -1 are the only values for which g is undefined and therefore the domain of g is $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

1.4 Manipulating Functions

The point of this section is see how one might manipulate and evaluate a given function, and thus enforce one's understanding of functions. The first example will be very important when defining derivatives, so please advise!

Example 1.4.1. Define the function f by $f(x) = \frac{1}{x+2}$. Note by the convention mentioned above that the domain of f is $(-\infty, -2) \cup (2, \infty)$ since only $f(-2)$ does not make sense. In any case, we will want to evaluate functions at values that are not explicitly chosen numbers. Say, for example, we want to see what $f(x+h)$ means. Well, in the same way we would input any old number, we just replace the 'x' in $f(x) = \frac{1}{x+2}$ exactly by '$x+h$', giving us

$$f(x+h) = \frac{1}{x+h+2}.$$

I really want to stress that we have simply just replace 'x' by '$x+h$' and that nothing fancy has occurred. (Strictly speaking, we need $x + h \neq -2$ since otherwise $f(x+h) = f(-2)$, which is not defined!)

Let's input some other values, including explicit and non-explicit numbers:

$$\begin{array}{ccc} f(a) = \frac{1}{a+2} & f(a^2) = \frac{1}{a^2+2} & f(a+b) = \frac{1}{a+b+2} \\ f(x-h) = \frac{1}{x-h+2} & f(-x) = \frac{1}{-x+2} & f(x+2h) = \frac{1}{x+2h+2} \\ f(2) = \frac{1}{4} & f(4) = \frac{1}{6} & f(-3) = -1. \end{array}$$

The next example will show us how to add, subtract, multiply, and so on, functions.

Example 1.4.2. Let $f(x) = x^2$ and $g(x) = x + 1$ be two functions. We note that we can understand what $f(x) \pm g(x)$, $\frac{f(x)}{g(x}$), and $f(x)g(x)$ all mean by simply inputing for x and then doing the given operation. Let's see examples of this.

$$\begin{array}{ccc} f(x)+g(x) = x^2+x+1 & f(2)+g(3) = 2^2+3+1 = 8 & f(-2)-g(-4) = (-2)^2-(-4+1) = 7 \\ \frac{f(x)}{g(x)} = \frac{x^2}{x+1} & \frac{f(2)}{g(3)} = \frac{2^2}{3+1} = 1 & \frac{f(-2)}{g(-4)} = \frac{(-2)^2}{-4+1} = -\frac{4}{3} \\ f(x)g(x) = x^2(x+1) & f(2)g(3) = 2^2(3+1) = 16 & f(-2)g(-4) = (-2)^2(-4+1) = -12 \end{array}$$

Our last example will explore something really important for defining derivatives and builds what we have already seen just above.

Example 1.4.3. Let $f(x) = x^2 + 1$. The following computation will be really useful (I am really stressing this for a reason!) in the near future.

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2+1-x^2-1}{h} = \frac{x^2+2xh+h^2-x^2-1}{h} = \frac{2xh+h^2}{h} = 2x+h.$$

Notice how the h has completely disappeared in the denominator! Take note again! it will be important.

1.5 Piecewise Functions

Example 1.5.1. Let us define a function f by the following two rules. On the set $[0,1]$, let $f(x) = x+1$, and on $(1,2)$, let $f(x) = x^3$. Note by the convention above, f has domain $[0,1] \cup (1,2) = [0,2)$.We see that we have defined the function f by two separate rules: one for $[0,1]$ and one for $(1,2)$. To express this with convenience notation, we write the following

$$f(x) = \begin{cases} x+1 & 0 \leq x \leq 1 \\ x^3 & 1 < x < 2 \end{cases}$$

So how do we use this notation? We shall call each line of this piecewise function a **case**. So, say we want to compute $f(1.5)$. We then find for which case does 1.5 fit in, which is the second case since $1 < 1.5 < 2$. Therefore, to compute $f(1.5)$, we use case two: $f(1.5) = 1.5^3 = 3.375$. Similarly, if we want to compute $f(0.5)$, we use case one since $0 < 0.5 \leq 1$ and so $f(0.5) = 0.5 + 1 = 1.5$. As a last point for clarification, note that $f(1) = 1 + 1 = 2$ and not $f(1) = 1^3 = 1$ since $0 < 1 \leq 1$.

As an extra point, we have that the range of f is $[1,2] \cup (1,8) = [1,8)$.

Example 1.5.2. Let us define a function g by the following three rules. On the set $[0,1]$, let $f(x) = x+1$; on $[1,2]$, let $f(x) = 2x$; and on $[2,3]$, let $f(x) = 4x$. Then, to express f as a piecewise function, we would write

$$f(x) = \begin{cases} x+1 & 0 \leq x \leq 1 \\ 2x & 1 \leq x \leq 2 \\ 4x & 2 < x < 3. \end{cases}$$

In preparation for future material, let us consider two examples that deal with piecewise functions with at least one case that is defined at a single value rather than an interval.

Example 1.5.3. Let h be a function defined by the following two rules. On the set $(-\infty,0) \cup (0,\infty)$, let $h(x) = \frac{1}{x}$; on the set containing only the number 0, let $h(x) = 1$. That is, h can be written as the piecewise function

$$h(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{x} & \text{otherwise.} \end{cases}$$

Note that we can also write h as

$$h(x) = \begin{cases} \frac{1}{x} & x < 0 \\ 1 & x = 0 \\ \frac{1}{x} & x > 0. \end{cases}$$

Example 1.5.4. Let k be the function defined by the following two rule. On the set $(-\infty,0) \cup (0,\infty)$, let $h(x) = \frac{1}{x}$. That is, h can be written as the piecewise function

$$k(x) = \begin{cases} \frac{1}{x} & x < 0 \\ \frac{1}{x} & x > 0. \end{cases}$$

This example shows us that it okay to define a piecewise function with a domain that is not $(-\infty,\infty)$; i.e., there may be "holes" in our function. Note that k in this example is different from h in the previous example.

1.6 The Absolute Value Function

The piecewise function considered in this section is important enough to warrant a name and will show up quite frequently. The function is called the **absolute value** function and is written as $f(x) = |x|$ (lo! another piece of notation!). This function is defined by the piecewise function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0; \end{cases}$$

i.e., $|x|$ is always a positive number, equal in magnitude to x. For example, $|-2| = 2$, $|-5| = 5$, $|2| = 2$, and so on.

1.7 Finding Domains of a Piecewise Function

In this section we show by example how to determine the domain of a piecewise function.

Example 1.7.1. *Let f be a function defined by*

$$f(x) = \begin{cases} x & 0 \le x < 5 \\ x^2 + 1 & x \ge 5 \end{cases}$$

Determine the domain of f in interval notation.

The trick here is to answer the question, Where is the function defined? What we shall do is analyze each case; namely, the $0 \le x < 5$ case and the $x \ge 5$ case. We see that on $0 \le x < 5$, $f(x) = x$ is defined everywhere here. Moreover, on $x \ge 5$, $f(x) = x^2 + 1$ is defined everywhere here. Lastly, we note that aside from on $0 \le x < 5$ and $x \ge 5$, the function is not defined whatsoever. Therefore, we may conclude that the domain of f is $[0, 5) \cup [5, \infty) = [0, \infty)$.

Note that what follows is purely conventional to this course and not something I agree with.

Example 1.7.2. *Let f be a function defined by*

$$f(x) = \begin{cases} \frac{1}{x^2+4x-12} & -10 \le x < 0 \\ x & x \ge 0. \end{cases}$$

Determine the domain of f in interval notation.

We again will analyze each case as we did above. The two cases to consider are $-10 \le x < 0$ and $x \ge 0$. In the first case, $f(x) = \frac{1}{x^2+4x-12}$, and so we see we risk dividing by zero. Thus, we need to check if $x^2 + 4x - 12 = 0$ when $-10 \le x < 0$ and note that if $x^2 + 4x - 12 = 0$ elsewhere, we don't have a problem with dividing by zero.

We may factor the polynomial to $x^2 + 4x - 12 = (x + 6)(x - 2)$, thus showing that $x^2 + 4x - 12 = 0$ only at $x = -6$ and $x = 2$. Thus, $x = -6$ and $x = 2$ are the potential bad points. However, on $x = -6$ satisfies $-10 \le x < 0$ and so, $x = -6$ is the only bad point for this function (to be sure, at $x = 2$, $f(x) = x = 2$). Thus, f is only defined on $-10 \le x < -6$, $-6 < x < 0$, and $x \ge 0$. It follows that the domain of f is $[-10, -6) \cup (-6, \infty)$.

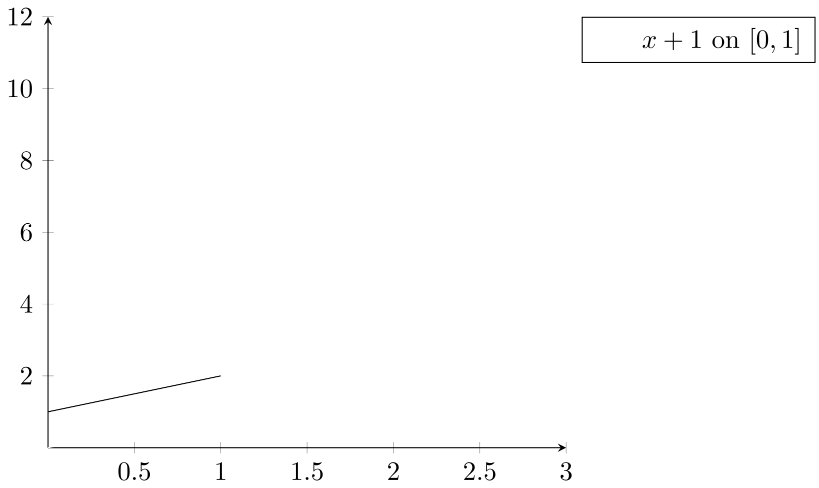
1.8 Graphing Piecewise Functions

To graph a piecewise function is essentially the way in which we normally graph a function, except now we graph each case of the piecewise function with respect to their respective intervals. There will be a convention to plotting piecewise graphs, however.

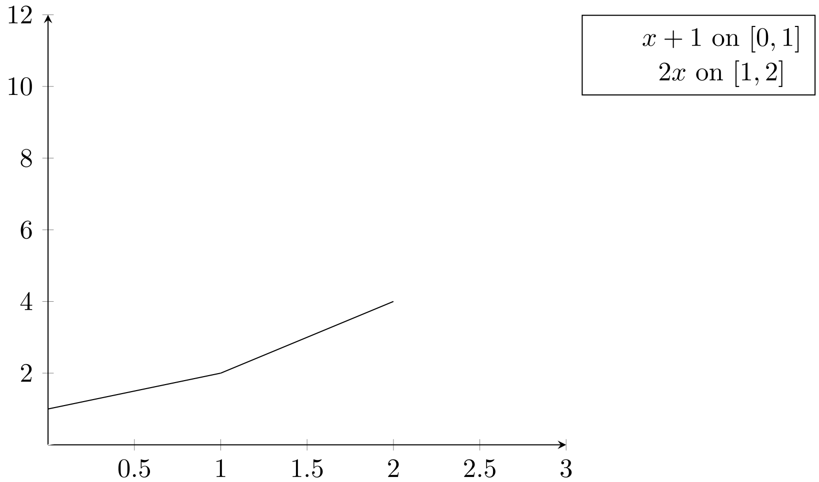
Convention. First draw bubbles at the end point of each case's domain. Second, if any cases are define at any of these endpoints, fill in the bubble at such a point. □

We shall show this and the method of graphing piecewise functions by example.

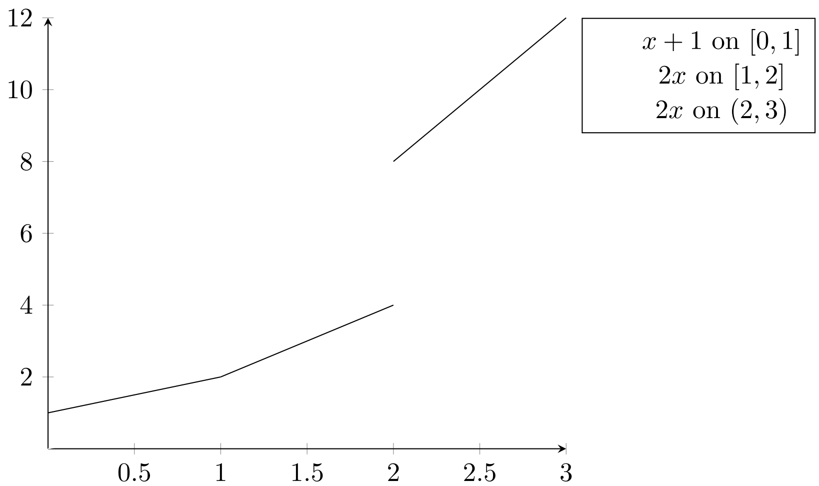
Example 1.8.1. Let h be the function in Example 1.5.2. We shall construct the graph of h step by step by considering each case of h and following the convention just mentioned. The first case of h is $x + 1$ on $[0, 1]$, so we must plot $x + 1$ on this interval.



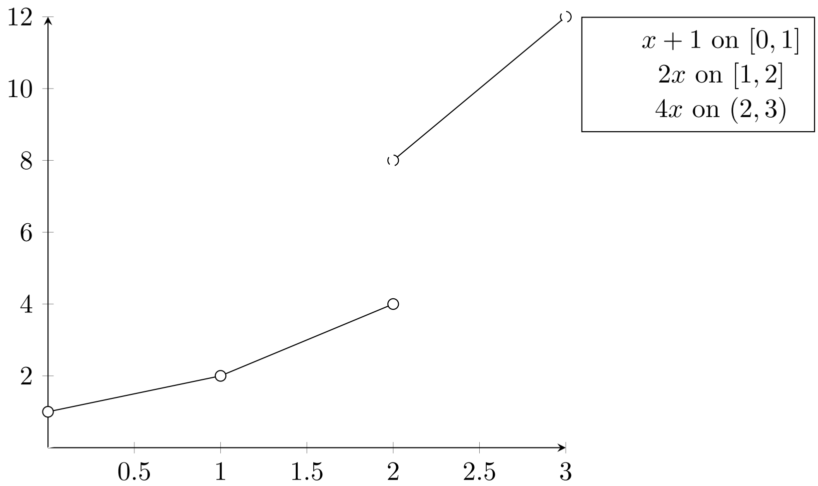
The second case of h is $2x$ on $[1, 2]$, so we must plot $2x$ on this interval.



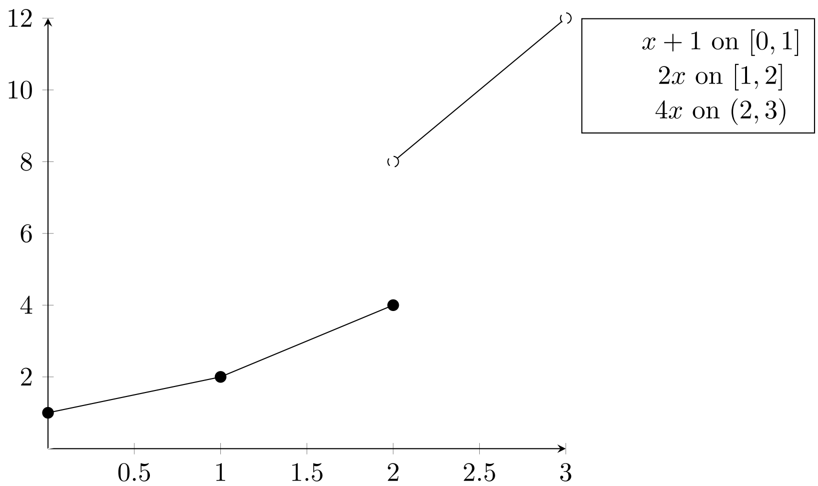
The last case of h is $4x$ on $(2, 3)$, so we must plot $4x$ on this interval.



We see that there is an ambiguity in these graphs with whether or not the points $(0, 1), (1, 2), (2, 4), (2, 8)$, or $(3, 12)$. Thus we must apply the just mentioned convention. We first draw open bubbles at all of these points since $0, 1, 2$ and 3 are end points of the cases' intervals.



But since $f(0), f(1)$ and $f(2)$ are defined, and so we must fill in the corresponding bubbles $(0, f(0)) = (0, 1)$, $(1, f(1)) = (1, 2)$, and $(2, f(2) = (2, 4)$. Moreover, since the case of f on $(2, 3)$ is not defined at 2 nor 3, we must leave these bubbles unfilled. We get the final graph.



1.9 Modeling with Piecewise Functions

The point of this section is to demonstrate how, when given a modeling type problem, how one can construct an appropriate piecewise function to model the situation.

Example 1.9.1. Suppose you run a business in need of some steel and that the steel supplier sells steel at \$800 per metric ton when purchasing up to 100 metric tons, and \$700 per metric ton when purchasing after 100 metric tons. Moreover, suppose the supplier requires you to pay a \$400 base service fee. Our task will be to model the cost of buying steel from this particular supplier.

Now, it is clear there are two cases to consider.If you wish to purchase steel at metric tons t in the interval $(0, 100]$, the steel will cost $\$800t + 400$, the \$800 coming from the price per metric tons, and the \$400 coming from the base service fee. Furthermore, if you wish to purchase steel at metrics tons t in the interval $(100, \infty)$, the steel will cost $\$700t + 400$. That is, if C is the cost function, then

$$C(t) = \begin{cases} 800t + 400 & 0 < t \leq 100 \\ 700t + 400 & 100 < t < \infty. \end{cases}$$