1 The Derivative

This section is where the calculus officially begins (the section on rates of change was perhaps an unofficial beginning). Let's see the definition now and give a discussion after.

Definition 1.0.1. If $y = f(x)$, the derivative of f at x (in the domain of f), denoted by $f'(x)$, y', $\frac{dy}{dx}$ and $\frac{d}{dx}f(x)$, is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided this limit exists. Moreover, if this limit exists at $x = c$, then f is said to be **differentiable** here.

The first thing we notice is that this coincides with the definition of instantaneous rate of change of a function at x. However, now we are dealing with a new function; viz., the derivative $f'(x)$ of a function f is itself another function who returns the instantaneous rate of change of f at x. This is important to remember as this is the whole point of the derivative. Calculus (and thus the derivative) was invented by Newton and others in hope of mathematically formalizing physics and other processes. The derivative's main importance was its ability to measure rates of change (e.g., it can give velocity, it can measure rate of income over time, etc.).

But this is not the whole story! The derivative is also very geometrical in that it allows one to approximate functions locally (and in fact linearly). That is, given a function, the derivative of the function allows us to analyze the function on some tiny interval (this is important for numerically analyzing functions or when the derivative is nice and the function is not or when we can only find the derivative but not the function).

But that's enough pseudo-justification for learning the derivative. How will we use it? We will learn more properties about derivative a little later, but for now, we stick with finding the derivative via the limit, tangent lines, consequences of the derivative, and when derivatives fail to exist.

1.1 Finding Derivatives via Limits

Here we shall do as we did before in finding instantaneous rates of change, as derivatives gives instantaneous rates of change. Now, however, the problems will be worded a bit differently.

Example 1.1.1 (Derivative of a constant)**.** Find the derivative of the function f defined by $f(x) = 5$ for all x. To do this, we apply the limit definition of the derivative. So, as before, we do the following simplification,

$$\frac{f(x+h) - f(x)}{h} = \frac{5 - 5}{h} = 0.$$

Thus, the derivative of f (remember, this is a function) is $f'(x) = 0$.

Example 1.1.2 (Derivative of x^2)**.** Let $f(x) = x^2$. We shall find its derivative $f'(x)$. This was done in the previous section, but let's revisit it with our new notation and vocabulary. First, we simplify,

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{2x+h}{.}$$

Thus, the derivative of f is given by

$$f'(x) = \lim_{h \to 0} (2x + h) = 2x.$$

That is, the derivative of f is given by the *function* $f'(x) = 2x$, and this function gives the instantaneous rate of change of f at x.

Example 1.1.3. *Find the derivative of $f(x) = x^2 + 5x$ using the limit definition of derivatives.*

We simply apply the definition of derivatives:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 5(x+h) - x^2 - 5x}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 5x + 5h - x^2 - 5x}{h} = \lim_{h \to 0} \frac{2xh + h^2 + 5h}{h} = 2x + 5.$$

1.2 Using the Derivative to find the Tangent Line

Like the previous section, this is pretty much just a rehash of what we did previously with tangent lines. The point here is to see that the derivative has its use in provided us with the slope of the tangent line in interest. That is, we can redefine the tangent line.

Definition 1.2.1. Let c be in the domain of a function f. Then the **tangent line** of f at $(c, f(c))$ has slope given by the limit

$$m_{tan} = f'(c)$$

provided the derivative exists, and it thus given by the equation

$$y_{tan} = m_{tan}(x - c) + f(c).$$

Let's see this in action.

Example 1.2.2 (Tangent line of x^2)**.** Let $f(x) = x^2$. We wish to find the tangent line of f at $x = 2$. Well, half the work was done in Example 1.1.2 since we have already found the derivative of f and since the derivative gives us the slope of the tangent line of interest. That is, the slope of the tangent line through $(2, f(2)) = (2, 4)$ is given by

$$m_{tan} = f'(2) = 4.$$

Then, the tangent line of f at $(2, 4)$ is given by, $y_{tan} = 4(x - 2) + 4 = 4x - 4$.

Example 1.2.3. *Find the derivative of* $f(x) = \frac{1}{x+1}$ *using the limit definition of derivatives, and find the equation of the tangent line at* $x = 2$*.*

We apply the definition of the derivative and then determine m_{tan} at $x = 2$. We find

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} = -\frac{1}{(x+1)^2}$$

after doing the appropriate algebra. It follows that $m_{tan} = f'(2) = -\frac{1}{(2+1)^2} = -\frac{1}{9}$. Thus, the equation of the tangent line is

$$y_{tan} = m_{tan}(x - 2) + f(2) = -\frac{1}{9}(x - 2) + \frac{1}{3}.$$

1.3 The Sign of the Derivative

In this section we discuss what the sign of the derivative can tell us about the function its the derivative of. We shall make use of the fact that the derivative gives the instantaneous rate of change of the function at some point. So, let f be the function of interest and f' its derivative. Then, $f'(c)$ gives the slope of the tangent line to the graph of f through $(c, f(c))$. Recall the discussion about the tangent line closely approximating the function. Then, we see that if $f'(c) > 0$, then the tangent line has a positive slope and is thus increasing. Therefore, since the tangent line closely approximates f, f too must be increasing at c (this will be made precise). Similarly, if $f'(c) < 0$, then the tangent has a negative slope and so it is decreasing. Therefore, f is decreasing at c. Lastly, if $f'(c) = 0$, then the tangent line is horizontal and so is not changing whatsoever (i.e., it is constant). Therefore, f is constant at c.

So what does we mean by saying a function f is increasing, decreasing, or constant at a point c? Well it turns out we kind of have to suspend belief and take the following definition,

Definition 1.3.1. Let f be a function, c in the domain of f, and y be the tangent line of f through $(c, f(c))$. Then,

- If y is increasing (has positive slope), f is increasing at c.
- If y is decreasing (has negative slope), f is decreasing at c.
- If y is constant (has zero slope), f is constant at c.

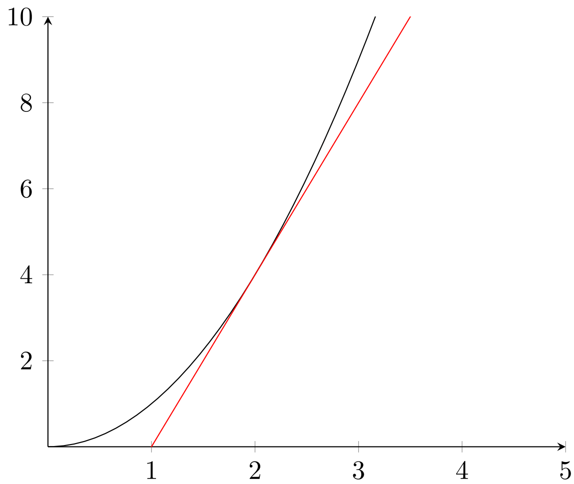

Figure 1: The function $f(x) = x^2$ increasing at $x = 2$

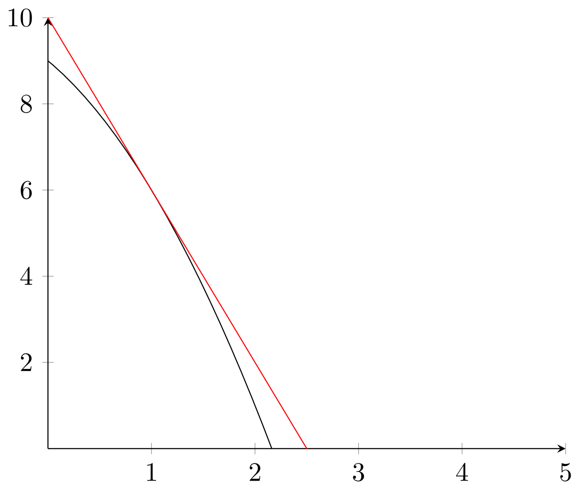

Figure 2: The function $g(x) = 10 - (x + 1)^2$ decreasing at $x = 1$

This may seem like a cop out (and it kind of is), but it is a very convenient way of defining these terms.

Let's see these concepts graphically. In Figure 1, we see the tangent line (red line) at $x = 2$ is increasing. It is also case that f is increasing (at least locally about c) around the point $x = 2$. In Figure 2, we see the tangent line (red line) at $x = 2$ is decreasing, and so is g. Lastly, in Figure 3, we see the tangent line at $x = 2$ is horizontal and so is constant. But it is hard to conclude whether or not h is constant here and so, after appealing to redefinition of constant, we must conclude h is constant at c. Let's summarize the implication of the sign of the derivative into the following theorem.

Theorem 1.3.2. *Let f be a function differentiable at c. Then*

1. *If $f'(c) < 0$, f is decreasing at c.*
2. *If $f'(c) = 0$, f is constant at c.*
3. *If $f'(c) > 0$, f is increasing at c.*

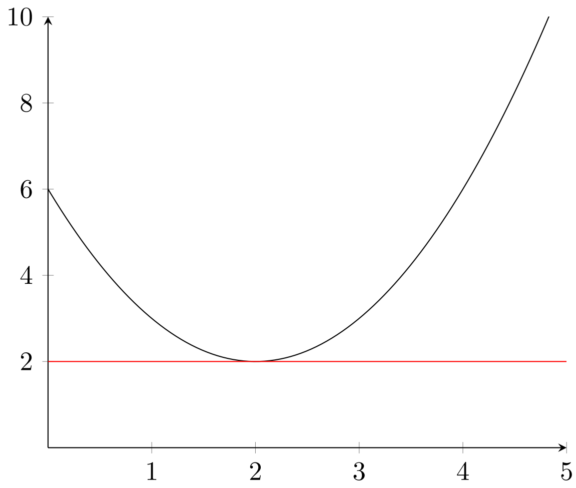

Figure 3: The function $h(x) = (x-2)^2 + 2$ constant at $x = 2$

1.4 Differentiability and Continuity

The following theorem is perhaps one that causes a lot of trouble. So, before I state it, we should all understand the following: if proposition P implies proposition Q, then it is not necessarily the case that Q implies P. For example, Being a dog implies having four legs, but having four legs does not imply being dog. Here, it turns out that differentiability implies continuity. That is, if f is differentiable at $x = c$, then we know that f is continuous at $x = c$. Moreover, regarding the warning, this does mean that f being continuous at $x = c$ implies f being differentiable at $x = c$. Let's state this in a theorem and see some examples.

Theorem 1.4.1. *Let f be differentiable at $x = c$. Then f is continuous at $x = c$.*

As a useful converse, we have the following.

Theorem 1.4.2. *Let f be discontinuous at $x = c$. Then f is not differentiable at $x = c$.*