The Second Derivative

We shall explore the second derivative of some function and what it may reveal to us about the function. When we speak of **the second derivative** we mean precisely differentiating the derivative of some function; i.e., differentiating the *second* time.

Definition (The Second Derivative)**.** Let f and f' be differentiable at x. Then the second derivative is given by

$$f''(x) = \frac{d^2}{dx^2} f(x) = \frac{d}{dx} f'(x).$$

Example. Find the second derivative of each of the following functions.

1. $f(x) = x^2$
2. $y = x^{-2} + e^x$
3. $G(x) = \ln(x)e^x$

Solution. In short, the procedure is to first find the functions $f'(x)$, $\frac{d}{dx}y(x)$, and $G'(x)$, and then differentiate these new functions to get $f''(x)$, $\frac{d^2}{dx^2}y$, and $G''(x)$.

1. We first find $f'(x) = \frac{d}{dx}x^2 = 2x$. Thus $f''(x) = \frac{d^2}{dx^2}x^2 = \frac{d}{dx}(2x) = x$.
2. We find $\frac{d}{dx}y = \frac{d}{dx}(x^{-2} + e^x) = -2x^{-3} + e^x$. It follows that $\frac{d^2}{dx^2}y = 6x^{-4} + e^x$.
3. Lastly,
$$G'(x) = x^{-1}e^x + \ln(x)e^x,$$
by the product rule. Thus
$$G''(x) = -x^{-2}e^x + x^{-1}e^x + x^{-1}e^x + \ln(x)e^x$$

The first piece of information f'' reveals to us about f is the notion of **concavity**.

Definition (Concavity)**.**

1. If $f''(x) > 0$ on (a, b) (i.e., if f' is increasing on (a, b)), then we say the graph of f is concave up on (a, b).
2. If $f''(x) < 0$ on (a, b) (i.e., if f' is decreasing on (a, b)), then we say the graph of f is concave down on (a, b).

The functions $f(x) = x^2$ and $g(x) = -x^2$ are the canonical examples of functions whose graphs are concave up and concave down, respectively. We see that $f''(x) = 2 > 0$ and $g''(x) = -2 < 0$ on any interval (a, b), which is enough to conclude f' is increasing on (a, b) and hence f has a concave up graph, and g' is decreasing on (a, b) and hence g has a concave down graph. However, what happens when the graph of a function goes from concave up to concave down or vice versa? We consider one last definition, that of the **inflection point**.

Definition.

1. If $f''(x) > 0$ on (a, c) and $f''(x) < 0$ on (c, b), then c is an inflection point.
2. If $f''(x) < 0$ on (a, c) and $f''(x) > 0$ on (c, b), then c is an inflection point.

Remark. Note that in case 1 we are saying that when the graph of f is concave up to the left of c and concave down to the right of c, then c is an inflection point. A similar remark can be made for case 2.

Example. Let $f(x) = x^3$. Note that $f'(x) = 3x^2$ and so $f'(x) = 0$ at $x = 0$. However, $f(0)$ is neither a maximum nor a minimum (as one can readily check with the graph of f). Furthermore, given $f''(x) = 6x$, we note $f''(x) > 0$ for $x > 0$ and $f''(x) < 0$ for $x < 0$. That is, on, say $(-1, 0)$, we have $f''(x) < 0$ and so the graph of f is concave down on $(-1, 0)$. Moreover, on, say, $(0, 1)$, we have $f''(x) > 0$ and so the graph of f is concave up on $(0, 1)$. Therefore, by the remark above, we conclude $x = 0$ is an inflection point of f and its *inflection value* is $f(0) = 0$.

We conclude this section with a theorem that is perhaps one of the more important facts the second derivative may reveal to us. This is the so called *Second Derivative Test*, which sometimes allows use to conclude when $f(c)$ is a relative minimum or a relative maximum if c is a critical point of f.

Theorem (The Second Derivative Test)**.** *Suppose $f'(c) = 0$. Then*

1. *If $f''(c) > 0$, then $f(c)$ is a relative minimum.*
2. *If $f''(c) < 0$, then $f(c)$ is a relative maximum.*
3. *If $f''(c) = 0$, then we cannot conclude anything.*

Remark. *Note that if $f''(c) = 0$ we are not completely at a loss. The first derivative test is still a valid option and should be used in this case; i.e., even if $f''(c) = 0$, $f(c)$ might be an extremal point and it might not.*

Example. Determine the relative extrema of the following functions.

1. $f(x) = 3x^2 + 2x + 1$
2. $y = x^2 e^x$
3. $H(x) = x^3$

Solution. The general procedure will be thus.

(a) Find the first derivatives $f'(x)$, y' and $H'(x)$.

(b) Find the critical points of the given functions.

(c) Find the second derivatives $f''(x)$, y'', and $H''(x)$.

(d) Use the second derivative test if conclusive, otherwise, determine if point is an inflection point or use the first derivative test.

We solve the example.

1. We find $f'(x) = 6x + 2$ and so, by $6x + 2 = 0$ when $x = -\frac{1}{3}$, we conclude $x = -\frac{1}{3}$ is a critical point of f. Next, f'' is given by $f''(x) = 6 > 0$. Note that $f''(-\frac{1}{3}) = 6$. We may thus conclude by the second derivative test that $f(-\frac{1}{3})$ is a relative minimum of f.

2. We find $y' = 2xe^x + x^2e^x$, and so
$$2xe^x + x^2e^x = xe^x(2 + x) = 0$$
only when $x = 0$ or $x = -2$; i.e., $x = 0$ and $x = -2$ are the critical points of y. Next we find
$$y'' = 2e^x + 2xe^2 + 2xe^x + x^2e^x.$$
All that is left to do is evaluate y'' at $x = 0$ and $x = -2$ and use the second derivative test. Evaluating y'' at $x = 0$, we find
$$y''(0) = 2e^0 + 2 \cdot 0e^0 + 2 \cdot 0e^0 + 0^2e^0 = 2 > 0,$$
and so $y(0)$ is a relative minimum of y. Next, evaluating y'' at $x = -2$, we find
$$y''(-2) = 2e^{-2} - 8e^{-2} + 4e^{-2} = -2e^{-2} < 0,$$
and so $y(-2)$ is a relative maximum of y.

3. Recall from above that H has a inflection point at $x = 0$. We shall show that the second derivative may fail to reveal anything here. Recall $H'(x) = 3x^2$ and find $H''(x) = 6x$. Note that $3x^2 = 0$ only at $x = 0$ and so H has the only critical point $x = 0$. However, $H''(0) = 6 \cdot 0 = 0$, and so the second derivative test may not be explicitly applied here. This is not to say that all is lost and that the second derivative is useless here! Note that $x = 0$ was the *only* critical point of H and so, finding that $H(0)$ is *not* a relative extrema, we may conclude H has no relative (and hence global) maxima nor minima.

Exercise. *Let $f(x) = x^4$. Show that the second derivative test may fail here, yet f has a local extremum at $x = 0$.*

Schemes for Problems

Finding the second derivative

Suppose you are given f.

1. First compute $f'(x)$.
2. Then compute $f''(x)$ by simply differentiating the new function $f'(x)$ obtained from differentiating f.

Finding where a function is concave up/down

Let f be a given function.

1. Find the second derivative f'' of f. Note that $f''(x)$ is just another function.
2. Find the intervals of continuity of f''.
3. Find the points where $f''(x) = 0$.
4. Construct intervals based off the information from 1 and 2. I can state this rigorously, but this will probably confuse things and so I think seeing this done by example is better. That is, refer to the example pertaining to this scheme.
5. Test the sign of f'' on each of these intervals and use the *Constant sign theorem*; i.e., if f'' is never zero on an interval and is continuous on said interval, then f has a constant sign here.
6. Apply definition of concavity.

Finding inflection points

Let f be a given function.

1. Find f''.
2. Find for what values of x does $f''(x) = 0$.
3. Suppose c is such a value for which $f''(c) = 0$. Compute $f''(u)$ and $f''(v)$ where $u < c < v$ and u and v are "very close" to c (again, better learned by example here).
4. Use definition of inflection point to conclude whether or not c is an inflection point.

Examples of the Schemes

Finding the second derivative

Refer to the example done in the notes. There isn't anything funky going on with finding the second derivative. You simply compute the first derivative f' of some function f, then compute the derivative f'' of this new function.

Finding where a function is concave up/down

Example. Find intervals for which the following functions are concave up or down.

(a) Let the second derivative of some function f be given by $f''(x) = \frac{(x+2)(x-2)}{(x-1)^4}$, and suppose only $x = 1$ is *not* in the domain of f.

(b) Let $g(x) = x^3 + x^2 + 1$.

Solution.

(a)
1. f'' is given and so we need not do anything.
2. Note that f'' is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$ since the only problem point is $x = 1$.
3. Note $f''(x) = 0$ only when $(x + 2)(x - 2) = 0$ since the denominator does not affect anything here. That is, $f''(x) = 0$ when $x = -2$ or $x = 2$.
4. The intervals of interest are going to be $(-\infty, -2), (-2, 1), (1, 2)$, and $(2, \infty)$ since on each of these intervals does f'' have a respective constant sign.
5. We test the sign of f'' for a point in each of the given intervals. We find the following results.
   - -3 is in $(-\infty, -2)$ and $f''(-3) = \frac{(-3+2)(-3-2)}{(-3-1)^4} = \frac{5}{256} > 0$ and so $f''(x) > 0$ on $(-\infty, -2)$ by the constant sign theorem. Therefore we may conclude that the graph of f is concave up on $(-\infty, -2)$.
   - 0 is in $(-2, 1)$ and $f''(0) = \frac{(0+2)(0-2)}{(0-1)^4} = -4 < 0$ and so $f''(x) < 0$ on $(-2, 1)$ by the constant sign theorem. Therefore the graph of f is concave down on $(-2, 1)$.
   - 1.5 is in $(1, 2)$ and $f''(1.5) = -28 < 0$ and so $f''(x) < 0$ on $(1, 2)$. Therefore the graph of f is concave down on $(1, 2)$.
   - 3 is in $(2, \infty)$ and $f''(3) = \frac{5}{16} > 0$ and so $f''(x) > 0$ on $(2, \infty)$. Therefore the graph of f is concave up on $(2, \infty)$.

Exercise. Use the second derivative test to determine the relative extrema.

(b)
1. $g'(x) = 3x^2 + 2x$ and so $g''(x) = 6x + 2$.
2. Note g'' is continuous everywhere.
3. $g''(x) = 6x + 2 = 0$ only when $x = -1/3$.
4. The intervals of interest are going to be $(-\infty, -1/3)$ and $(-1/3, \infty)$.
5. We test the sign of g'' on each of these intervals. We find the following results.
   - -1 is in $(-\infty, -1/3)$ and $g''(-1) = -1 \cdot 6 + 2 = -4 < 0$ and so $g''(x) < 0$ on $(-\infty, -1/3)$. Therefore the graph of g is concave down on $(-\infty, -1/3)$.
   - 0 is in $(-1/3, \infty)$ and $g''(0) = 2 > 0$ and so $g''(x) > 0$ on $(-1/3, \infty)$. Therefore the graph of g is concave up on $(-1/3, \infty)$.