Assignment 4

Due: 3/1/18 or 3/6/18

All relevant work must be shown in your solutions, even if it is not explicitly asked for you to explain.

In this problem, we will introduce the idea of matrix inversion (you can think of this as algebraically defining a notion of division). Let A be any $n \times n$ matrix (take note of the dimensions), and let I_n be the $n \times n$ identity matrix; i.e., I_n has ones along the diagonal and zeros everywhere else, and so $I_n A = AI_n = A$.

If there exists an $n \times n$ matrix B so that $AB = BA = I$, then B is said to be the matrix inverse of A, and we write $B = A^{-1}$. Note that A^{-1} exists if and only if $\det(A) \neq 0$, and in this case we say A is invertible or nonsingular.[1] Note that if A is invertible, then the system $Ax = b$ can be solved by simply applying A^{-1} to both sides:

$$x = A^{-1}Ax = A^{-1}b.$$

In the following problem, we compute the matrix inverse of a 2x2 and a 3x3. In general, if A is invertible, then placing the augmented matrix $(A|I_n)$ into rref produces the augmented matrix $(I_n|A^{-1})$.

Example. Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

Then the augmented matrix $(A|I_2)$ takes the form

$$(A|I_2) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}.$$

We place $(A|I_2)$ into rref:

$$\begin{aligned} (A|I_2) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix} &\xrightarrow{-2R_1+R_2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{pmatrix} \\ &\xrightarrow{-R_2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{pmatrix} \xrightarrow{-2R_2+R_1} \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{pmatrix} = (I_2|A^{-1}), \end{aligned}$$

and so

$$A^{-1} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}.$$

[1] Hence, algebraically, the collection of all $n \times n$ matrices is not closed under inversion. The matrices with $\det(A) = 0$ have a sort of degeneracy, and cause problems in both pure and applied mathematics. Interestingly, when solving systems $Ax = b$ for x, if $\det(A)$ is very nearly zero, A is "approximately" degenerate, and solving the system is computationally more exhausting.

Problem 1

(i) Compute the respective determinants of the following two matrices.

(ii) Compute the respective inverses of the following two matrices.

(iii) After you have computed A^{-1} in (a), show explicitly $A^{-1}A = I_2$ by multiplying out the two matrices by hand.

To show your work: First choose a method of computing the determinant by hand, and then do it out explicitly. Second, find the inverse of the matrix as detailed in the example above (for the 2x2, you may use the formula if you know it). Third, when finding the inverse, please list every elementary row operation you used.

(a)

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}.$$

(b)

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 2 & 3 \\ 9 & 3 & 5 \end{pmatrix}.$$