Problem 1

(a) $\tan(y)$ is very nonlinear! so the ODE is nonlinear.
the highest order derivative is $y' \Rightarrow$ order = 1.

(b) Check $y=0$ solves the ODE:

$y=0 \Rightarrow y'=0$.
$\tan 0 = 0$.

Hence

$$x \cdot y' = \tan y \Rightarrow x \cdot 0 = \tan 0 \Rightarrow 0=0$$

So $y=0$ satisfies the ODE!

(c) $xy' = \tan y \rightarrow \dfrac{dy}{\tan y} = \dfrac{dx}{x}$

$$\downarrow$$

$$\int \cot y \, dy = \int \frac{dx}{x}$$

$\downarrow$ (by hint)

$$\ln(\sin(x)) = \ln x + C$$

$$\sin y = e^{\ln x + C} = Ae^{\ln x} = Ax$$

$$\boxed{y = \sin^{-1}(Ax).}$$

Problem 2

We check: $y' = \cos x + \sin(x)$

$y'' = -\sin(x) + \cos x$

ODE $\quad y'' + y = -\sin(x) + \cos x + \sin(x) - \cos x + 2 = 2$ ✓

Initial values: $y(0) = \sin(0) - \cos(0) + 2 = 1$ ✓

$y'(0) = \cos 0 + \sin 0 = 1$ ✓

Problem 3

Use $u = x+y+1$ (or $u = x+y$). Then $\frac{d}{dx}u = \frac{d}{dx}x + \frac{dy}{dx} \rightarrow u' = 1 + y'$.

Hence

$$y' = \sqrt{x+y+1} + x + y \rightarrow u' - 1 = \sqrt{u} + x + y \rightarrow u' = \sqrt{u} + x + y + 1 = \sqrt{u} + u.$$

So

$$\frac{du}{\sqrt{u}+u} = dx \rightarrow \int \frac{du}{\sqrt{u}+u} = \int dx \rightarrow 2\ln(\sqrt{u}+1) = x + C.$$

Solving for u: $\quad u = (Ae^{x/2} - 1)^2$

Solving for y: $\quad u = x+y+1 \Rightarrow y = u - x - 1 = (Ae^{x/2} - 1)^2 - x - 1.$