Problem 1

We compute the Wronskian at $t=0$:

$$W(X_1(0), X_2(0), X_3(0)) = \det\begin{pmatrix} 1 & 1 & 2 \\ 6 & -2 & 3 \\ -13 & -1 & -2 \end{pmatrix} = 4 - 39 - 12 - 52 + 3 + 12 \neq 0$$

$$\begin{array}{ccccc} 1 & 1 & 2 & 1 & 1 \\ 6 & -2 & 3 & 6 & -2 \\ -13 & -1 & -2 & -13 & -1 \end{array}$$

So X_1, X_2, X_3 are linearly independent. Moreover, there are 3 vectors, and A is $3\times 3 \Rightarrow$ fundamental set.

Problem 2

We take

$$X_1(t) = \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t}, \quad X_2(t) = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t}, \quad X_3(t) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t}$$

We should show X_1, X_2, X_3 each satisfy

$$X'(t) = A X(t).$$

Lets just check this for $X_1(t)$:

$$X_1'(t) = \begin{pmatrix} -6 \\ 1 \\ 5 \end{pmatrix} e^{-t}$$

$$A X_1(t) = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t} = \begin{pmatrix} -6 \\ 1 \\ 5 \end{pmatrix} e^{-t} = X_1'(t)$$

So X_1 is a solution.

We compute the Wronskian at $t=0$:

$$W(X_1(0), X_2(0), X_3(0)) = \det\begin{pmatrix} 6 & -3 & 2 \\ -1 & 1 & 1 \\ -5 & 1 & 1 \end{pmatrix} = 20 \neq 0$$

So X_1, X_2, X_3 are linearly independent. Since A is 3×3, they form a fundamental set.

Hence

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + c_3 X_3(t)$$

is the general solution.