Assignment 7

Due: 4/26/18

All relevant work must be shown in your solutions, even if it is not explicitly asked for you to explain.

Problem 1

(3pts)

Let $f(t)$ be a periodic function of period $T > 0$; i.e., $f(t+T) = f(t)$. Show that

$$\mathscr{L}\{f(t)\}(s) = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}dt.$$

(We assume f is nice enough for everything to work.) You should effectively follow these steps:

1. First use the fact that we may write the decomposition

$$\int_0^\infty F(x)dx = \sum_{j=0}^{\infty} \int_{jT}^{(j+1)T} F(x)dx$$

   for nice enough functions F.

2. Use the change of variable $t - jT = x$ for the jth integral.
3. Use the exponent property $e^{a+b} = e^a e^b$, and then factor out the appropriate exponent from the respective integral.
4. Use periodicity of f to write each integral in the summation as $\int_0^T f(x)e^{-sx}dx$.
5. Since each integral in the summation is the same, factor it out from the summation.
6. Lastly, conclude the proof by applying the power series formula to the summation.

To receive credit, you must write everything in a neat and clear logical order; use complete sentences. Otherwise it will be graded to be a zero.

Problem 2

(3pts)

Define f on the interval $0 \leq t < 1$ by $f(t) = t$. Suppose f is extended periodically with period 1 to $0 \leq t < \infty$, thus obtaining the sawtooth signal; i.e., f is extended so that $f(t+1) = f(t)$ for $0 \leq t < \infty$. Compute $\mathscr{L}\{f(t)\}$. Use Problem 1 and compute the integral explicitly showing each step.

Problem 1

$$\mathscr{L}\{f(t)\}(s) = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}dt.$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \sum_{j=0}^\infty \int_{jT}^{(j+1)T} f(t)e^{-st}dt$$

$$= \sum_{j=0}^\infty \int_0^T f(u+jT)e^{-su-sjT}du \qquad u=t-jT$$

$$= \sum_{j=0}^\infty e^{-sjT}\int_0^T f(u)e^{-su}du \qquad \text{by periodicity of } f$$

$$= \int_0^T f(u)e^{-su}du \sum_{j=0}^\infty e^{-sjT}$$

$$= \int_0^T f(u)e^{-su}du \; \frac{1}{1-e^{-sT}} \quad \text{as desired.}$$

Problem 2

$T=1$

$f(t)=t$ on $0\le t<1$.

Then by 1)

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-s}}\int_0^1 te^{-st}dt = \frac{1}{1-e^{-s}}\left[-\frac{1}{s^2}e^{-st}-\frac{1}{s}te^{-st}\right]_0^1.$$

$$(t\,e^{-st})' = e^{-st}-ste^{-st} \rightarrow te^{-st} = \frac{1}{s}e^{-st}-\frac{1}{s}(te^{-st})'$$
$$= \left(-\frac{1}{s^2}e^{-st}-\frac{1}{s}te^{-st}\right)'$$