3.2 Nonlinear Models

- Idea: Somethings exhibit nonlinear behavior.

Ex Logistic Equation: $\frac{dP}{dt} = P(a-bP)$.

P = population
$\frac{dP}{dt}$ = population growth

Features: when $P = \frac{a}{b}$, $\frac{dP}{dt} = 0$.

$\therefore$

a) $\frac{dP}{dt} > 0$ when $P < \frac{b}{a}$ $\rightarrow$ growth
b) $\frac{dP}{dt} = 0$ when $P = \frac{b}{a}$ $\rightarrow$ no growth
c) $\frac{dP}{dt} < 0$ when $P > \frac{b}{a}$ $\rightarrow$ decay

Solution P:

Solve $\frac{dP}{dt} = P(a-bP)$:

$$\frac{dP}{P(a-bP)} = \left(\frac{1/a}{P} + \frac{b/a}{a-bP}\right) dP = dt$$

$$\therefore \quad \frac{1}{a} \ln P - \frac{1}{a} \ln (a-bP) = t + C$$

$$\therefore \quad \ln \frac{P}{a-bP} = at + C$$

$$\therefore \quad \frac{P}{a-bP} = ce^{at}$$

$$\therefore \quad P = \frac{ac}{bc + e^{-at}}$$

Suppose $P(0) = P_0$

$\therefore$

$$P(t) = \frac{aP_0}{bP_0 + (a-bP_0)e^{-at}}$$

Long Term Behavior:

$$\lim_{t \to \infty} \frac{aP_0}{bP_0 + (a-bP_0)e^{-at}} = \frac{a}{b}.$$

$\therefore$ The model $\frac{dP}{dt} = P(a-bP)$ stabilizes to $\frac{a}{b}$ as $t \to \infty$.

We call $\frac{a}{b}$ the limiting value.

<u>Ex 1</u>

- limiting number $= \frac{a}{b} = 100 \rightarrow a = 1000b$
- $P(0) = P_0 = 10$
- $P(1) = 100$

$\therefore$

$P(t) =$

$$100 = P(1) = \frac{10a}{10b + (a - 10b)e^{-a\cdot 1}} = \frac{10\cdot 1000b}{10\cdot b + (1000b - 10b)e^{-1000b\cdot 1}}$$

$\hookrightarrow \quad b \cong 0.0023$

$\hookrightarrow \quad a \cong 2.3$

$$\therefore \quad P(t) \cong \frac{23}{0.023 + (2.3 - 0.023)e^{-2.3t}}$$