Exam 2 Review

Section 5.1: Driven Motion

A mass of 1 slug is attached to a spring whose constant is 5lb/ft. Initially, the mass is released 1 foot below the equilibrium position with downward velocity of 5ft/s, and the surrounding medium offers a damping force numerically equal to 2 times the instantaneous velocity. Find the equation of motion if the mass is driven by an external force of $f(t) = 12\cos 2t + 3\sin 2t$.

Solution.

The goal is to set up and solve the IVP

$$\begin{cases} mx'' + \beta x' + kx = f(t) \\ x(0) = x_0 \\ x'(0) = x_1 \end{cases}.$$

We first translate everything into mathematical variables and quantities:

$$\begin{aligned} m &= 1 \\ k &= 5 \\ x(0) &= 1 \\ x'(0) &= 5 \\ \beta &= 2 \\ f(t) &= 12\cos 2t + 3\sin 2t \end{aligned}$$

Thus we need to solve the IVP

$$\begin{cases} x'' + 2x' + 5x = 12\cos 2t + 3\sin 2t \\ x(0) = 1 \\ x'(0) = 5 \end{cases}.$$

First we solve the associated homogeneous equation

$$x'' + 2x' + 5x = 0.$$

The appropriate auxiliary equation is

$$n^2 + 2n + 5 = 0,$$

where I am using n in place of m to not confuse with m as the mass. Using the quadratic formula, we get

$$
\begin{aligned}
n &= \frac{-2 \pm \sqrt{4-20}}{2} \\
&= \frac{-2 \pm \sqrt{-16}}{2} \\
&= \frac{-2 \pm \sqrt{-1}\sqrt{-16}}{2} \\
&= \frac{-2 \pm 4i}{2} \\
&= -1 \pm 2i.
\end{aligned}
$$

Recall that if the roots to the auxiliary equation are $n = \alpha \pm \beta i$, then the corresponding complementary function $x_c(t)$ is given by

$$x_c(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t.$$

Since our roots are $n = -1 \pm 2i$, we have $\alpha = -1$, and $\beta = 2$. Therefore, the complementary function to the given ODE is

$$x_c(t) = c_1 e^{-t} \sin 2t + c_2 e^{-t} \sin 2t.$$

Now we need to solve the nonhomogeneous ODE

$$x'' + 2x' + 5x = 12 \cos 2t + 3 \sin 2t$$

using either undetermined coefficients (4.4) or variation of parameters (4.6). We will use undetermined coefficients here.

Since

$$f(t) = 12 \cos 2t + 3 \sin 2t,$$

we make the initial guess that

$$x_p(t) = A \cos 2t + B \sin 2t.$$

Note that x_p and x_c do not share any functions of the exact same type ($e^{-t} \sin 2t$ and $\sin 2t$, or $e^{-t} \cos 2t$ and $\cos 2t$ are not considered as the same type). Therefore, there is no duplication and hence no need to multiply by t to remove duplication.

We compute

$$
\begin{aligned}
x_p(t) &= A \cos 2t + B \sin 2t \\
x_p'(t) &= -2A \sin 2t + 2B \cos 2t \\
x_p''(t) &= -4A \cos 2t - 4B \sin 2t.
\end{aligned}
$$

Therefore, after plugging in x_p, x_p', x_p'' into the given nonhomogeneous ODE, we arrive at

$$
\begin{aligned}
12 \cos 2t + 3 \sin 2t &= x_p'' + 2x_p' + 5x_p \\
&= -4A \cos 2t - 4B \sin 2t - 4A \sin 2t + 4B \cos 2t + 5A \cos 2t + 5B \sin 2t \\
&= (-4A + 4B + 5A) \cos 2t + (-4B - 4A + 5B) \sin 2t \\
&= (4B + A) \cos 2t + (B - 4A) \sin 2t.
\end{aligned}
$$

Comparing coefficients, we get

$$\begin{aligned} 12 &= 4B + A \\ 3 &= B - 4A. \end{aligned}$$

Solving this system, we conclude $A = 0$ and $B = 3$.

Therefore

$$x_p(t) = 3\sin 2t,$$

and so the general solution to the ODE is

$$x(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + 3\sin 2t.$$

Now use $x(0) = 1$, $x'(0) = 5$ to get that $c_1 = 1$ and $c_2 = 0$ (sorry, I really don't feel like typing that out...).

Section 4.6: Variation of Parameters

Recall

$$\begin{aligned} ay'' + by' + cy &= f(x) \\ y_c &= c_1 y_1 + c_2 y_2 \\ y_p &= u_1 y_1 + u_2 y_2 \\ u_1' &= \frac{-y_2 f}{W(y_1, y_2)} \\ u_2' &= \frac{y_1 f}{W(y_1, y_2)} \end{aligned}$$

Solve $y'' + 5y' + 6y = e^x$.

Solution

First we solve the associated homogeneous equation

$$y'' + 5y' + 6y = 0.$$

The auxiliary equation is

$$m^2 + 5m + 6 = (m+2)(m+3) = 0.$$

Therefore, the complementary solution is

$$y_c = c_1 e^{-2x} + c_2 e^{-3x} = c_1 y_1 + c_2 y_2.$$

By the variation of parameters method, the particular solution should take the form

$$y_p = u_1 y_1 + u_2 y_2.$$

Note $f(x) = e^x$. We compute the Wronskian first

$$W(y_1, y_2) = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = e^{-2x}(-3e^{-3x}) - (-2e^{-2x}e^{-3x}) = -e^{-5x}.$$

Therefore

$$
\begin{aligned}
u_1' &= -\frac{y_2 f}{W(y_1, y_2)} = -\frac{e^{-3x}e^{x}}{-e^{-5x}} = e^{3x} \\
u_2' &= \frac{y_1 f}{W(y_1, y_2)} = \frac{e^{-2x}e^{x}}{-e^{-5x}} = -e^{4x}.
\end{aligned}
$$

Therefore, by integration, we get (recall we can omit the constant of integration here)

$$
\begin{aligned}
u_1 &= \int e^{3x} dx = \frac{1}{3}e^{3x} \\
u_2 &= \int -e^{4x} dx = -\frac{1}{4}e^{4x}.
\end{aligned}
$$

It follows that

$$y_p = u_1 y_1 + u_2 y_2,$$

and so the general solution is

$$y = y_c + y_p.$$

Section 3.3: Mixing Problem and Circuits Problems

Given in class.

Sections 3.1, 3.2

Worry about qualitative problems.

Logistics equation:

$$\begin{aligned} \frac{dP}{dt} &= P(a - bP) \\ \lim_{t\to\infty} P(t) &= \frac{a}{b} \end{aligned}.$$

How does the system behave if $P(0) > a/b$ or $P(0) < a/b$ or $P(0) = a/b$.

Idea: Note that if $P(0) > a/b$, then $bP(0) - a > 0$ or $a - bP(0) < 0$. Assume $P(0) > 0$. It follows that at $t = 0$, $P' = P(a - bP) < 0$. Therefore, the population will experience growth since its rate of change is positive. A similar analysis holds for the other cases. In particular, if $P(0) < a/b$, then the population will experience decay, and if $P(0) = a/b$, the population will not experience change (this can be directly seen from the solution).

Temperature equation:

$$\frac{dT}{dt} = k(T - T_m).$$

How does the system behave if $T(0) > T_m$ or $T(0) < T_m$ or $T(0) = T_m$?

Idea: A similar analysis from above holds. However, it will depend on the sign of k chosen. E.g., if $T > T_m$, and $k > 0$, then $k(T - T_m) > 0$, and so T' will be positive, and therefore T will increase.