4.6 Variation of Parameters

$y'' + P(x)y' + Q(x)y = f(x)$

$y_c = c_1 y_1 + c_2 y_2$ ← Complementary function

$y_p = u_1(x) y_1 + u_2(x) y_2$ ← particular solution

$u_1' = \frac{-y_2 f(x)}{W(y_1, y_2)}$, $u_2' = \frac{y_1 f(x)}{W(y_1, y_2)}$

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ ← Wronskian

Then gen. sol. is

$$y = y_c + y_p$$

Ex

$y'' - 4y' + 4y = (x+1)e^{2x}$

1) $y'' - 4y' + 4y = 0 \rightarrow y_c = c_1 e^{2x} + c_2 x e^{2x} \rightarrow y_1 = e^{2x}, y_2 = xe^{2x}$.

2) $y_p = u_1 e^{2x} + u_2 x e^{2x}$, $f(x) = (x+1)e^{2x}$

$W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = 2xe^{4x} + e^{4x} - 2xe^{4x} = e^{4x}$

$u_1' = -\frac{y_2 f(x)}{W(y_1, y_2)} = -\frac{xe^{2x} \cdot (x+1)e^{2x}}{e^{4x}} = -x^2 - x \rightarrow u_1 = -\frac{x^3}{3} - \frac{x^2}{2}$

$u_2' = \frac{y_1 f(x)}{W(y_1, y_2)} = \frac{e^{2x}(x+1)e^{2x}}{e^{4x}} = x+1 \rightarrow u_2 = \frac{x^2}{2} + x$

$\therefore \quad y_p = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)x e^{2x}$

3) $y = y_c + y_p$.

Ex

$4y'' + 36y = \csc 3x$

Standard form: $y'' + 9y = \frac{1}{4} \csc 3x$

1) $y'' + 9y = 0 \rightarrow y_c = c_1 \cos 3x + c_2 \sin 3x \rightarrow y_1 = \cos 3x$, $y_2 = \sin 3x$

2) $y_p = u_1 y_1 + u_2 y_2$, $f(x) = \frac{1}{4} \csc 3x$

$W(y_1, y_2) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$

$u_1' = -\frac{y_2 f(x)}{W(y_1, y_2)} = \frac{-\sin 3x \cdot \frac{1}{4} \csc 3x}{3} = -\frac{1}{12} \rightarrow u_1 = -\frac{1}{12}x$

$u_2' = \frac{y_1 f(x)}{W(y_1, y_2)} = \frac{\cos 3x \frac{1}{4} \csc 3x}{3} = \frac{1}{12} \cot 3x \rightarrow u_2 = \frac{1}{36} \ln|\sin 3x|$

$\therefore \quad y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{12}\cos 3x + \frac{1}{36}\ln|\sin 3x|\sin 3x$

3) $\therefore \quad y = y_c + y_p$.