Visiting an Old Friend: Calc II

In this note we will review the calc II that will be used quite frequently in the study of the Laplace transformation.

Integration by Parts

Recall the product rule

$$(uv)' = u'v + uv'.$$

It follows by the fundamental theorem of calculus that, after integrating both sides, we get

$$uv\Big|_a^b = \int_a^b \frac{d}{dx}(uv)dx = \int_a^b \frac{du}{dx}vdx + \int_a^b u\frac{dv}{dx}dx.$$

Therefore, we have the more familiar formula:

$$\int_a^b \frac{du}{dx}vdx = uv\Big|_a^b - \int_a^b u\frac{dv}{dx}dx.$$

At some point, it becomes more useful to think of integration by parts as a formula for moving the derivative operator $\frac{d}{dx}$ from one function to the other all while under an integral sign.

Example 1.

$$\begin{aligned}\int_a^b e^{-2x}xdx &= \int_a^b \frac{d}{dx}\left(-\frac{1}{2}e^{-2x}\right)xdx \\ &= -\frac{1}{2}e^{-2x}x\Big|_a^b - \int_a^b -\frac{1}{2}e^{-2x}\frac{dx}{dx}dx \\ &= -\frac{1}{2}e^{-2x}x\Big|_a^b + \int_a^b e^{2x}dx,\end{aligned}$$

where $u = -\frac{1}{2}e^{-2x}$ and $v = x$.

Partial Fractions

There are only really two types of partial fractions we will have to worry about. However, I am not saying these will be the only two cases, but rather any partial fractions you will see will basically fit into these two schemes. Honestly, in practice, either look up the formula, or make an educated guess and check if it works (that's what I would do if I wasn't ripping off the textbook's examples).

Example 2.

This will be the simplest case where the denominator has unique factors:

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \left(\frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}\right).$$

Now we multiply through by $(s-1)(s-2)(s+4)$, then plug in clever values for s, and then solve for A, B, C.

First, after multiplying, we get

$$s^2+6s+9 = (s-2)(s+4)A + (s-1)(s+4)B + (s-1)(s-2)C.$$

Plugging in $s = 2$ will make two of the terms vanish:

$$25 = 0 \times A + 6B + 0 \times C = 6B,$$

so that $B = \frac{25}{6}$.

Plugging in $s = 1$ will make another two terms vanish:

$$16 = -5A + 0 \times B + 0 \times C = -5A,$$

so that $A = -\frac{16}{5}$

Lastly, plugging in $s = -4$, we get

$$1 = 0 \times A + 0 \times B + 30C = 30C$$

so so that $C = \frac{1}{30}$.

Therefore,

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}.$$

Example 3

The rule of thumb to remember is that if a factor $(s-a)^n$ is in the denominator, then your partial fraction expansion should include the n partial fractions with dominators $(s-a), (s-a)^2, \ldots, (s-a)^n$. E.g., in the following example, we take $n = 3$:

$$\frac{2s+5}{(s-3)^3} = \frac{A}{s-3} + \frac{B}{(s-3)^2} + \frac{C}{(s-3)^3}.$$

Multiplying through by $(s-3)^3$ results in

$$2s+5 = A(s-3)^2 + B(s-3) + C.$$

We will compare coefficients on both sides of the equation to determine A, B, C, and so we expand the right hand side:

$$2s+5 = As^2 - 6As + 9A + Bs - 3B + C,$$

so that

$$\begin{aligned}
s^2 &: \qquad A = 0 \\
s &: \qquad -6A + B = 2 \\
1 &: \qquad A - 3B + C = 5
\end{aligned}$$

It follows that $A = 0$, $B = 2$, and $C = 11$. Therefore,

$$\frac{2s+5}{(s-3)^3} = \frac{2}{(s-3)^2} + \frac{11}{(s-3)^3}.$$

Improper Integrals

The only improper integrals we will consider are of the form

$$\int_a^\infty f(t)dt.$$

So all you need to recall is that

$$\int_a^\infty f(t)dt = \lim_{b\to\infty} \int_a^b f(t)dt.$$

Let $p(t)$ denote any polynomial. Important limits are of the form (thus expect slight variations):

$$\begin{aligned}
\lim_{t\to\infty} e^{-t}p(t) &= 0 \\
\lim_{t\to\infty} e^{t}p(t) &= \pm\infty \\
\lim_{t\to\infty} e^{-t}\cos(t) &= 0.
\end{aligned}$$

The middle limit is $\pm\infty$ as opposed to just $+\infty$ because $p(t)$ might tend to $-\infty$ as $t \to \infty$ (e.g., $p(t) = -t$).

Example 4.

$$\begin{aligned}
\int_0^\infty e^{-t}dt &= \lim_{b\to\infty} \int_0^b e^{-t}dt \\
&= \lim_{b\to\infty} -e^{-t}\Big|_0^b \\
&= \lim_{b\to\infty} \left(-e^{-b} + e^0\right) \\
&= 0 + 1 \\
&= 1.
\end{aligned}$$