Some Hyperbolic Trigonometry

In this note we will briefly explore the hyperbolic trig functions. By "elliptic trig," I will mean the usual trig functions you are familiar with. By elliptic geometry, one usually means the geometry of a circle (or (hyper)sphere), and so, since the usual trig functions are intimately tied to the unit circle, this terminology makes sense. In the same way, hyperbolic trig functions are tied to "hyperbolic geometry." Actually, hyperbolic trig functions are intimately tied to the "unit hyperbola" in an almost exact analogous way to the way elliptic trig is tied to the unit circle. (If you want to go down the hyperbolic geometric rabbit hole, look into the connection between hyperbolic geometry and M. C. Escher's art.)

Hyperbolic functions

$$\begin{aligned}
\cosh(x) &= \frac{e^x + e^{-x}}{2} \qquad \text{(read as "cosh of x")} \\
\sinh(x) &= \frac{e^x - e^{-x}}{2} \qquad \text{(read as "sinch of x")} \\
\tanh(x) &= \frac{\sinh(x)}{\cosh(x)} \qquad \text{(read as "tinch of x").}
\end{aligned}$$

Other expressions like hyperbolic cosecant, hyperbolic secant, etc., are analogous to the elliptic trig function versions; e.g., $\operatorname{sech}(x) = \frac{1}{\cosh(x)}$. If you feel silly saying "tinch" etc., it is okay to say "hyperbolic tangent" and so on.

Differentiation and integration

$$\begin{aligned}
\frac{d}{dx}\cosh(x) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{e^x - e^{-x}}{2} \\
&= \sinh(x) \\
\frac{d}{dx}\sinh(x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\
&= \frac{e^x + e^{-x}}{2} \\
&= \cosh(x) \\
\int \cosh(x)dx &= \sinh(x) + C \\
\int \sinh(x)dx &= \cosh(x) + C.
\end{aligned}$$

Note that the derivative formulas do not pick up an extra negative sign like the elliptic trig functions would.

Limits, Special Values, and Identities

$$\begin{aligned}
\lim_{x\to\pm\infty}\cosh(x) &= \lim_{x\to\pm\infty}\frac{e^x+e^{-x}}{2} = \frac{e^{\pm\infty}+e^{-\pm\infty}}{2} = \infty \\
\lim_{x\to\infty}\sinh(x) &= \lim_{x\to\infty}\frac{e^x-e^{-x}}{2} = \frac{e^{\infty}-e^{-\infty}}{2} = \infty \\
\lim_{x\to-\infty}\sinh(x) &= \lim_{x\to-\infty}\frac{e^x-e^{-x}}{2} = \frac{e^{-\infty}-e^{\infty}}{2} = -\infty \\
\cosh(0) &= \frac{e^0+e^0}{2} = 1 \\
\sinh(0) &= \frac{e^0-e^0}{2} = 0 \\
\cosh^2(x)-\sinh^2(x) &= 1 \\
\cosh(-x) &= \cosh(x) \\
\sinh(-x) &= -\sinh(x).
\end{aligned}$$

Note that hyperbolic trig is *not* periodic. However, just as periodicity is the essence of elliptic geometry, unboundedness is the essence of hyperbolic geometry (as evidenced by the limits above).

Connection to Elliptic Trig

Recall Euler's formula:

$$e^{it} = \cos(t) + i\sin(t).$$

It follows that

$$\begin{aligned}
\frac{e^{it}+e^{-it}}{2} &= \frac{\cos(t)+i\sin(t)+\cos(t)-i\sin(t)}{2} = \cos(t) \\
\frac{e^{it}-e^{-it}}{2i} &= \frac{\cos(t)+i\sin(t)-\cos(t)+i\sin(t)}{2i} = \sin(t).
\end{aligned}$$

Therefore, by the change of variable $x \mapsto ix$, we have

$$\begin{aligned}
\cos(ix) &= \frac{e^{i\cdot ix}+e^{-i\cdot ix}}{2} = \frac{e^{-x}+e^{x}}{2} = \cosh(x) \\
\sin(ix) &= \frac{e^{i\cdot ix}-e^{-i\cdot ix}}{2i} = \frac{e^{-ix}-e^{ix}}{2i} = -\frac{1}{i}\frac{e^x-e^{-x}}{2} = -\frac{1}{i}\sinh(x) = i\sinh(x).
\end{aligned}$$

This change of variable is not superficial–there is some deep mathematical theory behind why elliptic geometry is somehow "dual" to hyperbolic geometry. Unfortunately, it'd probably take a few years of studying to understand this.