An Introduction to Systems of Differential Equations

Joshua Flynn

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Chapter 1

Introduction

In this note we will learn how to solve systems of differential equations of the form

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) \\ x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t), \end{aligned}$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are real constants, and $x_1(t), x_2(t)$ are the unknowns we wish to find. These systems of differential equations model evolving coupled systems.

Chapter 2

Matrix Algebra

In this section we briefly review 2×2 matrix algebra. Note that everything that follows has a $n \times n$ analogue–if you're interested in this theory, ask me, or take linear algebra. Henceforth, let $\mathbb{R}$ denote the real numbers, and $\mathbb{C}$ denote the complex numbers. (Actually, everything that follows also has a generalization over "number fields" different than the $\mathbb{R}$ and $\mathbb{C}$.)

2.1 Matrices and Vectors

The following two algebraic objects are what we will mostly deal with:

$$\begin{array}{ll} 2 \times 2 \text{ Matrix:} & A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ 2 \times 1 \text{ Vector:} & x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{array}.$$

Here, a, b, c, d and x_1, x_2 are in $\mathbb{R}$. Thus, a 2×2 matrix is just a 2×2 array of 4 numbers, and a vector is just a column of 2 numbers.

Note that a 2×2 matrix may also be given with double indices:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

There is a distinguished matrix called the **identity matrix**:

$$\text{Identity Matrix:} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

There is also a distinguished vector called the **zero vector**:

$$\text{Zero Vector:} \quad 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Be careful: 0 *might either indicate the zero vector or the number zero–context should make it clear which zero is being considered.*

2.2 Multiplication by a Scalar

Let λ be in $\mathbb{R}$ or $\mathbb{C}$. Then

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}, \qquad \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}.$$

Example 2.1.

$$\begin{pmatrix} 8 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

△

2.3 Matrix acting on a vector

$$Ax = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}.$$

In particular, the identity matrix $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$ acts like the number 1 in terms of multiplication:

$$Ix = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1x_1 + 0x_2 \\ 0x_1 + 1x_2 \end{pmatrix} = x.$$

Moreover, $A0 = 0$ for all matrices A:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a0 + b0 \\ c0 + d0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Example 2.2. Let

$$A = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}, \qquad x = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Then

$$Ax = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 + 1 \cdot 4 \\ 0 \cdot 2 + -2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \end{pmatrix}.$$

△

2.4 Linearity of this Action

$$A(c_1 x + c_2 y) = c_1 Ax + c_2 Ay$$

for all 2×1 vectors x and y, and all numbers c_1, c_2.

2.5 Matrix Addition

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

be two matrices. Then, their summation $A + B$ is defined by

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}.$$

Similarly, two vectors may be added:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}.$$

Example 2.3.

$$\begin{pmatrix} -2 & 3 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 3 & -7 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -2+3 & 3-7 \\ -2+2 & 4+0 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 0 & 4 \end{pmatrix}.$$

△

Example 2.4.

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix} + \begin{pmatrix} -7 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-7 \\ -4+1 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$$

△

2.6 Determinant of a matrix

$$\det(A) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Example 2.5. Let

$$A = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}.$$

Then,

$$\det(A) = \det \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} = 2 \cdot (-2) - 1 \cdot 0 = -4.$$

△

2.7 Solving a Matrix Equation

Given the known data

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

we will be interested in solving equations like

$$Ax = b,$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is to be determined (i.e., solved for). By matrix multiplication, the equation $Ax = b$ is equivalent to

$$\begin{aligned} Ax = b \iff & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ \iff & \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ \iff & \begin{matrix} ax_1 + bx_2 = b_1 \\ cx_1 + dx_2 = b_2 \end{matrix}. \end{aligned}$$

Therefore, solving for x in $Ax = b$ is equivalent to solving for x_1 and x_2 in the system

$$\begin{aligned} ax_1 + bx_2 &= b_1 \\ cx_1 + dx_2 &= b_2. \end{aligned}$$

In particular, we get the translation between a system of equations to a matrix equation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \iff \begin{matrix} ax_1 + bx_2 = b_1 \\ cx_1 + dx_2 = b_2. \end{matrix}$$

Example 2.6. Solve the equation

$$\begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

First we convert this matrix equation into a system of equations:

$$\begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \iff \begin{matrix} 2x_1 + 3x_2 = 2 \\ 2x_1 + 2x_2 = 6. \end{matrix}$$

We use elimination to solve for x_1 and x_2:

$$\begin{aligned} \begin{matrix} 2x_1 + 3x_2 = 2 \\ 2x_1 + 2x_2 = 6 \end{matrix} \iff & \begin{matrix} 2x_1 + 3x_2 = 2 \\ 0x_1 - 1x_2 = 4 \end{matrix} \\ \iff & \begin{matrix} 2x_1 + 0x_2 = 14 \\ 0x_1 - 1x_2 = 4 \end{matrix} \\ \iff & \begin{matrix} 1x_1 + 0x_2 = 7 \\ 0x_1 + 1x_2 = -4 \end{matrix} \\ \iff & \begin{matrix} x_1 = 7 \\ x_2 = -4 \end{matrix}. \end{aligned}$$

Therefore, the solution is

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}.$$

△

Example 2.7. Solve

$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

First we convert this matrix equation into a system of equations:

$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{aligned} 2x_1 - 1x_2 &= 0 \\ -4x_1 + 2x_2 &= 0 \end{aligned}$$

$$\iff \begin{aligned} 2x_1 - 1x_2 &= 0 \\ 0x_1 + 0x_2 &= 0 \end{aligned}$$

$$\iff x_1 = \frac{1}{2}x_2.$$

In this case, we cannot actually find unique values for x_1 and x_2, but rather, x_1 and x_2 satisfy the relationship $x_1 = \frac{1}{2}x_2$. Consequently, we get infinitely many solutions: for a chosen x_2, we get a corresponding x_1.

Therefore, the solution is written as

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix},$$

where x_2 is allowed to take on any real value; i.e., there are infinitely many solutions. E.g., to get what we mean by this, the following are particular examples of solutions:

$$\begin{aligned} x_2 = 1 &\implies x = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \\ x_2 = 0 &\implies x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ x_2 = -5 &\implies x = \begin{pmatrix} -5 \\ -\frac{5}{2} \end{pmatrix}. \end{aligned}$$

△

Example 2.8. Convert the following system of equations into a matrix equation:

$$\begin{aligned} 8x_1 - 2x_2 &= 0 \\ -7x_1 + 0.2x_2 &= 2. \end{aligned}$$

The coefficients produce the following matrix:

$$\begin{aligned} 8x_1 - 2x_2 &= 0 \\ -7x_1 + 0.2x_2 &= 2 \end{aligned} \iff A = \begin{pmatrix} 8 & -2 \\ -7 & 0.2 \end{pmatrix}.$$

The variables x_1 and x_2 are encoded in the vector

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and the constants not attached to any variables are encoded by the vector

$$b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Putting this together, we get the translation:

$$\begin{aligned} 8x_1 - 2x_2 &= 0 \\ -7x_1 + 0.2x_2 &= 2. \end{aligned} \iff \begin{pmatrix} 8 & -2 \\ -7 & 0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

△

2.8 Linear Dependence

The following practically only works in the 2×1 vector setting, but there is a natural extension to the higher dimensional setting. This generalization is important in case you want to study higher dimensional systems of DEs; however, we will focus only on the two dimensional case, and so the following will be sufficient.

Let v and u be two 2×1 vectors. Then v and u are said to be **linearly dependent** provided there is a nonzero constant c so that

$$u = cv;$$

i.e., provided that u is a scalar multiple of v. Otherwise, u and v are said to be **linearly independent**.

Example 2.9. *Determine if*

$$v = \begin{pmatrix} 2 \\ 8 \end{pmatrix}, \qquad u = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$$

are linearly dependent.

We seek for a nonzero constant c such that $u = cv$; i.e.,

$$\begin{pmatrix} 2 \\ 8 \end{pmatrix} = c \begin{pmatrix} -1 \\ -4 \end{pmatrix} = \begin{pmatrix} -c \\ -4c \end{pmatrix}$$

Thus,

$$\begin{aligned} 2 &= -c \\ 8 &= -4c \end{aligned} \iff \begin{aligned} c &= -2 \\ c &= -2' \end{aligned}$$

so that $c = -2$ works. Therefore, $u = -2v$ holds, and hence u and v are linearly dependent. △

Example 2.10. *Determine if*

$$v = \begin{pmatrix} 2 \\ 8 \end{pmatrix}, \qquad u = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

are linearly dependent.

We seek for a nonzero constant c such that $u = cv$; i.e.,

$$\begin{pmatrix} 2 \\ 8 \end{pmatrix} = c \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} c \\ -4c \end{pmatrix}.$$

Thus,

$$\begin{aligned} 2 &= c \\ 8 &= -4c \end{aligned} \iff \begin{aligned} c &= 2 \\ c &= -2 \end{aligned},$$

which is clearly a contradiction: c cannot simultaneously be 2 and -2. Therefore, u and v are linearly independent. △

2.9 Eigenvalue problem

2.9.1 Eigendata Derivation

Given $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, find all pairs λ, x, where λ is in $\mathbb{C}$, and $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ is a nonzero vector, satisfying

$$Ax = \lambda x.$$

Such a λ is called an **eigenvalue**, and such a x is called an **eigenvector**. Therefore, the eigenvalue problem is to solve for x and λ in the equation $Ax = \lambda x$; i.e., λ and x are to be understood as unknowns to be found. I reiterate an important point: *the zero vector* $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ *is not allowed to be called an eigenvector.*

Note that the eigenvalues and eigenvectors which solve $Ax = \lambda x$ for a given matrix A are intrinsic to A. They comprise the sort of "spirit" of A.

The following properties will be needed to be understood:

1. If $Ax = \lambda x$, then cx also satisfies this equation for any constant c:

$$A(cx) = c(Ax) = c\lambda x = \lambda(cx).$$

2. A can have either (i) 2 distinct real eigenvalues, (ii) 2 distinct complex eigenvalues, or (iii) 1 eigenvalue with multiplicity 2.

For our course in DEs, you can translate these two properties as follows:

1. Your answer might disagree with someone else's, but only up to a constant multiple.

2. In cases (i) and (ii), you will have to find two distinct eigenvectors which are not constant multiples of each other. In case (iii), you'll have to do something special.

The following requires some understanding of linear algebra; however, you can get by by black-boxing the following linear algebra fact (here A and x are arbitrary):

The matrix equation $Ax = 0$ has a nonzero solution x if and only if $\det(A) = 0$.

We now solve the eigenvalue problem $Ax = \lambda x$. Observe:

$$\begin{aligned} Ax = \lambda x &\iff Ax = \lambda I x \\ &\iff Ax - \lambda I x = 0 \\ &\iff (A - \lambda I)x = 0. \end{aligned}$$

Therefore, there is a nonzero solution x (i.e., an eigenvector) if and only if

$$\det(A - \lambda I) = 0.$$

So how do we actually go about finding λ and x? Well, observe

$$A - \lambda I = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix},$$

and so

$$\begin{aligned} \det(A - \lambda I) = 0 &\iff \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0 \\ &\iff (a-\lambda)(d-\lambda) - bc = 0 \\ &\iff \lambda^2 - (a+d)\lambda + ad - bc = 0 \\ &\iff \lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4\det A}}{2} \end{aligned}$$

Therefore, eigenvalues must solve the **characteristic equation**

$$\lambda^2 - (a+d)\lambda + ad - bc = 0,$$

and are given by the roots

$$\begin{aligned} \lambda_1 &= \frac{(a+d) + \sqrt{(a+d)^2 - 4\det A}}{2} \\ \lambda_2 &= \frac{(a+d) - \sqrt{(a+d)^2 - 4\det A}}{2}. \end{aligned}$$

To find x, we need to solve the matrix equation $(A - \lambda I)x = 0$. Equivalently, since we know that solving a matrix equation may be done by solving its associated system of equation, we need to solve the following system:

$$\begin{aligned} (A - \lambda I)x = 0 &\iff \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\iff \begin{matrix} (a-\lambda)x_1 + bx_2 = 0 \\ cx_1 + (d-\lambda)x_2 = 0 \end{matrix} \\ &\iff \begin{matrix} x_1 = \dfrac{b}{\lambda - a} x_2 \\ x_1 = \dfrac{\lambda - d}{c} x_2, \end{matrix} \end{aligned}$$

where the last equivalence holds only if we do not divide by zero. Thus, to find an eigenvector, it is enough to solve this system of equations.

2.9.2 Eigendata Examples

Example 2.11. *Verify that*

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

are eigenvectors of

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

What are their corresponding eigenvalues?

We first compute

$$Av_1 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+4 \\ 2+8 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5v_1,$$

showing that v_1 is an eigenvector of A with eigenvalue $\lambda_1 = 5$. Next, we compute

$$Av_2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+2 \\ -4+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 0v_2,$$

showing that v_2 is an eigenvector of A with eigenvalue $\lambda_2 = 0$. △

Example 2.12. *Find the eigenvalues and eigenvectors of*

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}.$$

Eigenvalues:

$$\begin{aligned} \det A &= 4 - 2 = 2 \\ \lambda_1 &= \frac{(a+d) + \sqrt{(a+d)^2 - 4\det A}}{2} \\ &= \frac{4 + \sqrt{16-8}}{2} \\ &= 2 + \sqrt{2} \\ \lambda_2 &= \frac{(a+d) - \sqrt{(a+d)^2 - 4\det A}}{2} \\ &= 2 - \sqrt{2}. \end{aligned}$$

△

Eigenvectors:

For $\lambda_1 = 2 + \sqrt{2}$:

$$\begin{aligned} (a-\lambda_1)x_1 + bx_2 = 0 \\ cx_1 + (d-\lambda_1)x_2 = 0 \end{aligned} \iff \begin{aligned} -\sqrt{2}x_1 + x_2 = 0 \\ 2x_1 - \sqrt{2}x_2 = 0 \end{aligned}$$

$$\iff x_1 = \frac{1}{\sqrt{2}} x_2$$

Taking $x_2 = 1$, we have $x_1 = \frac{1}{\sqrt{2}}$, and therefore

$$k_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to λ_1.

For $\lambda_2 = 2 - \sqrt{2}$:

$$\begin{aligned} (a-\lambda_2)x_1 + bx_2 = 0 \\ cx_1 + (d-\lambda_2)x_2 = 0 \end{aligned} \iff \begin{aligned} \sqrt{2}x_1 + x_2 = 0 \\ 2x_1 + \sqrt{2}x_2 = 0 \end{aligned}$$

$$\iff x_1 = -\frac{1}{\sqrt{2}}x_2$$

Taking $x_2 = 1$, we have $x_1 = -\frac{1}{\sqrt{2}}$, and therefore

$$k_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to λ_2.

Example 2.13. *Find the eigenvalues and eigenvectors of*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Eigenvalues:

$$\begin{aligned} \det A &= 2(3) - 0(0) = 6 \\ \lambda_1 &= \frac{5 + \sqrt{25 - 24}2}{=}2 \\ \lambda_2 &= 3 \end{aligned}$$

Eigenvectors:
For $\lambda_1 = 2$:

$$\begin{aligned} (a-\lambda_1)x_1 + bx_2 = 0 \\ cx_1 + (d-\lambda_1)x_2 = 0 \end{aligned} \iff \begin{aligned} 0x_1 + 0x_2 = 0 \\ 0x_1 + x_2 = 0. \end{aligned}$$

This shows that x_1 can be any nonzero value (since we are looking for eigenvectors), and $x_2 = 0$. Therefore, taking $x_1 = 1$, we get

$$k_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_1 = 2$.
For $\lambda_2 = 6$:

$$\begin{aligned} (a-\lambda_1)x_1 + bx_2 = 0 \\ cx_1 + (d-\lambda_1)x_2 = 0 \end{aligned} \iff \begin{aligned} -x_1 + 0x_2 = 0 \\ 0x_1 + 0x_2 = 0. \end{aligned}$$

This shows that x_2 can be any nonzero value, and $x_1 = 0$. Therefore, taking $x_2 = 1$, we get

$$k_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_2 = 3$.

△

2.10 Problems

Problem 2.1. *Compute the following matrix operations:*

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} = ?$

(b) $\begin{pmatrix} 2 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \pi \end{pmatrix} = ?$

(c) $\begin{pmatrix} 2 & 3 \\ 6 & 1 \end{pmatrix} \left(5 \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right) = ?$

Problem 2.2. *Compute the determinant of the following matrices:*

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ -4 & 100 \end{pmatrix}$

Problem 2.3. *Find the eigenvalues and eigenvectors of the following matrices:*

(a) $\begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Problem 2.4. *What are the eigenvalues and eigenvectors of the matrices:*

(a) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

(b) $B = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$

Remark: in case $a = b$, you can either use inspection, or try to expand on what we have learned.

Chapter 3

Systems of Differential Equations

Consider the following system:

Insert Image here...

where $x_A(t), x_B(t)$ are amounts of some quantity, and $f_i(x_A), g_i(x_B)$ are the input or output rates of the quantity. Note that $f_i(x_A), g_i(x_B)$ are to be understood as depending on the quantities x_A or x_B, not just time.

So the subsystems A and B, and the quantities x_A and x_B, are all "coupled" as reflected in the following system of DEs:

$$\begin{aligned} x'_A(t) &= (\text{Rate in of quantity into } A) - (\text{Rate out of quantity out of } A) \\ &= f_1(x_A) + g_1(x_B) - f_2(x_A) \\ x'_B(t) &= (\text{Rate in of quantity into } B) - (\text{Rate out of quantity out of } B) \\ &= f_2(x_A) - g_1(x_B) - g_2(x_B). \end{aligned}$$

Our goal is to use linear algebraic methods to solve such "coupled" systems of DEs.

Note: everything can be generalized to more equations and variables, which would correspond to a system with more coupled subsystems. However, this has little pedagogical value, especially since knowledge of linear algebra is not assumed. In any case, the only difference between our setting and the more general setting is a heavier use of linear algebra. If a time where you need to study larger systems comes, you will likely have more linear algebra under your belt, and you can then generalize as needed.

3.1 A System of DEs

We wish to solve the following system of DEs:

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) \\ x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t), \end{aligned} \tag{3.1}$$

where the coefficients a_{ij} are real constants, and the functions $x_1(t), x_2(t)$ are the unknowns we wish to solve.

The key observation is that we can recast this system of a matrix differential equation. Observe that the right-hand side of (3.1) may be encoded by the following matrix product:

$$AX(t) := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \iff \begin{matrix} a_{11}x_1(t) + a_{12}x_2(t) \\ a_{21}x_1(t) + a_{22}x_2(t), \end{matrix}$$

where we will use a capital $X(t)$ to denote the vector-valued function $\left(\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}\right)$. Therefore, (3.1) translates into

$$\begin{aligned} \begin{matrix} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) \end{matrix} &\iff \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ &\iff X'(t) = AX(t), \end{aligned}$$

where, for convenience, we use the notation $X'(t)$ to represent the vector

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = X'(t).$$

3.2 Solution Vectors

Let $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$. If the vector $X(t) = \left(\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}\right)$ satisfies the matrix DE $X'(t) = AX(t)$ on an interval I, then $X(t)$ is called a **solution vector** to this matrix DE on I.

Note: a solution vector $X(t) = \left(\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}\right)$ is such that the functions $x_1(t), x_2(t)$ solves the following system of DEs:

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) \\ x_2'(t) &+ a_{21}x_1(t) + a_{22}x_2(t). \end{aligned}$$

Indeed, we have the follow logical equivalences

$$\begin{aligned} X'(t) = AX(t) &\iff \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ &\iff \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a_{11}x_1(t) + a_{12}x_2(t) \\ a_{21}x_1(t) + a_{22}x_2(t) \end{pmatrix} \\ &\iff \begin{matrix} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) \\ x_2'(t) + a_{21}x_1(t) + a_{22}x_2(t). \end{matrix} \end{aligned}$$

Example 3.1. *Verify that*

$$X(t) = \begin{pmatrix} e^{2t} \\ -7e^{3t} \end{pmatrix}$$

is a solution to the matrix DE

$$X'(t) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} X(t).$$

What corresponding system do the components of $X(t)$ solve?

We are given

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad X(t) = \begin{pmatrix} e^{2t} \\ -7e^{3t} \end{pmatrix}.$$

All that we are missing from $X'(t) = AX(t)$ is $X'(t)$, and so we compute:

$$X'(t) = \frac{d}{dt} \begin{pmatrix} e^{2t} \\ -7e^{3t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -21e^{3t} \end{pmatrix}.$$

Now we check if $X'(t) = AX(t)$ holds by computing $AX(t)$:

$$AX(t) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} e^{2t} \\ -7e^{3t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -21e^{3t} \end{pmatrix}.$$

But the right-hand side is exactly $X'(t)$, i.e., $X'(t) = AX(t)$ is a true statement, and therefore $X(t) = \begin{pmatrix} e^{2t} \\ -7e^{3t} \end{pmatrix}$ is a solution vector.

Lastly, we aim to determine which system the functions $x_1(t) = e^{2t}, x_2(t) = -7e^{3t}$ (i.e,. the components of $X(t)$) solve. This amounts to translating the matrix DE $X'(t) = AX(t)$ into a system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \iff \begin{aligned} x_1'(t) &= 2x_1(t) + 0x_2(t) = 2x_1(t) \\ x_2'(t) &= 0x_1(t) + 3x_2(t) = 3x_2(t). \end{aligned}$$

△

3.3 Superposition Principle

There is a systems analogue of the superposition principle we are familiar with from studying linear ODEs.

Theorem. *Let $X_1(t), X_2(t)$ be two solutions to the matrix DE $X'(t) = AX(t)$ on an interval I. Then, for all real numbers c_1, c_2, $c_1X_1(t) + c_2X_2(t)$ is also a solution on I.*

Proof.

1. By hypothesis, $X_1'(t) = AX_1(t)$ and $X_2'(t) = AX_2(t)$.

2. To show that $c_1X_1(t) + c_2X_2(t)$ is a solution to $X'(t) = AX(t)$, we need to verify that

$$(c_1X_1(t) + c_2X_2(t))' = A(c_1X_1(t) + c_2X_2(t)).$$

3. We compute:

$$(c_1X_1(t)+c_2X_2(t))' = c_1X_1'(t)+c_2X_2'(t) \qquad \text{(distribution of differentiation of sums)}$$
$$A(c_1X_1(t)+c_2X_2(t)) = c_1AX_1(t)+c_2AX_2(t) \qquad \text{(linearity of matrix multiplication)}$$

4. By 1. and 3.,

$$\begin{aligned}(c_1X_1(t)+c_2X_2(t))' &= c_1X_1'(t)+c_2X_2'(t)\\ &= c_1(AX_1(t))+c_2(AX_2(t))\\ &= c_1AX_1(t)+c_2AX_2(t)\\ &= A(c_1X_1(t)+c_2X_2(t)),\end{aligned}$$

which is what we needed to show by 2.

□

Example 3.2. *Verify*

$$X_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}, \qquad X_2(t) = \begin{pmatrix} 0 \\ e^{3t} \end{pmatrix}$$

are both solutions to the matrix DE

$$X'(t) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} X(t).$$

Conclude that $c_1X_1(t)+c_2X_2(t)$ is also a solution. Compare this to Example 3.1.

1. We compute directly that $X_1(t)$ and $X_2(t)$ are solutions by plugging them into $X'(t) = AX(t)$ and checking if equality holds:

$$\begin{aligned} X_1'(t) = AX_1(t) &\iff \begin{pmatrix} 2e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}\\ &\iff \begin{pmatrix} 2e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 0 \end{pmatrix}\\ X_2'(t) = AX_2(t) &\iff \begin{pmatrix} 0 \\ 3e^{3t} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ e^{3t} \end{pmatrix}\\ &\iff \begin{pmatrix} 0 \\ 3e^{3t} \end{pmatrix} = \begin{pmatrix} 0 \\ 3e^{3t} \end{pmatrix},\end{aligned}$$

showing that equality holds in both cases, i.e., that $X_1(t)$ and $X_2(t)$ are solutions.

2. By the superposition principle, we may conclude that $c_1X_1(t)+c_2X_2(t)$ is a solution for any real constants c_1 and c_2.

3. Observe that the matrix DE in this Example and Example 3.1 are the same. Moreover, the solution given in Example 3.1 is given by

$$X(t) = \begin{pmatrix} e^{2t} \\ -7e^{3t} \end{pmatrix}.$$

We observe that

$$\begin{pmatrix} e^{2t} \\ -7e^{3t} \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{3t} \end{pmatrix}$$

provided $c_1 = 1$ and $c_2 = -7$. Thus the solution in Example 3.1 is a particular example from step 2.

△

3.4 Linear Independent Solutions

Recall from Chapter 1 that two constant vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

are linearly dependent if they there is a nonzero c such that $x = cy$, i.e., if x is a nonzero scalar multiple of y. This notion is extended to solutions to a matrix DE $X'(t) = AX(t)$ in an analogous way:

> *If $X_1(t)$ and $X_2(t)$ are solutions to $X'(t) = AX(t)$ on an interval I, then they are called* ***linearly dependent solutions*** *if there is a nonzero constant c such that $X_1(t) = cX_2(t)$ for all t in I.*

More generally, two vector-valued functions $X_1(t), X_2(t)$ (not necessarily solution vectors) are called linearly dependent on an interval I if and only if $X_1(t) = cX_2(t)$ for all t in I.

In practice, this will be relatively easy in our course. As we will see, all solutions will take the form of a linear combination of solution vectors like

$$X(t) = \begin{pmatrix} a_1 e^{\lambda t} \\ a_2 e^{\lambda t} \end{pmatrix},$$

where a_1, a_2 are constants. The key observation is that $e^{\lambda t} = ce^{\beta t}$ for some constant c if and only if $\lambda = \beta$. Indeed,

$$e^{\lambda t} = ce^{\beta t} \implies e^{(\lambda-\beta)t} = c$$

is impossible if $\lambda \neq \beta$ since the left-hand side would then not be constant. This is observed in the following proposition.

Proposition 3.3. *Consider the two vector-valued functions:*

$$X_1(t) = K_1 e^{\lambda_1 t} = \begin{pmatrix} k_{11} \\ k_{21} \end{pmatrix} e^{\lambda_1}, \qquad X_2(t) = K_2 e^{\lambda_2 t} = \begin{pmatrix} k_{21} \\ k_{22} \end{pmatrix} e^{\lambda_2 t}.$$

If $K_1 \neq K_2$ or $\lambda_1 \neq \lambda_2$, then $X_1(t)$ and $X_2(t)$ are linearly independent.

Example 3.4. *The two vector-valued functions*

$$X_1(t) = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}, \qquad X_2(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

are solutions to the system

$$X'(t) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X(t).$$

Are they linearly independent? Here,

$$\begin{aligned} X_1(t) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}, \qquad X_2(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \\ K_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \lambda_1 &= 3, \qquad \lambda_2 = -1. \end{aligned}$$

Since $K_1 \neq K_2$, they are linearly independent. This can also be concluded from $\lambda_1 \neq \lambda_2$. △

3.5 The General Solution

Given a matrix DE $X'(t) = AX(t)$ with associated system

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) \\ x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t), \end{aligned}$$

one should expect a general solution to involve a two parameter family of solutions, one parameter arising from "integrating away" each derivative, namely x_1' and x_2'.

The following theorem confirms this.

Theorem 3.5. *Let $X_1(t), X_2(t)$ be two linearly independent solutions to $X'(t) = AX(t)$ on an interval I. Then the general solution to $X'(t) = AX(t)$ is given by*

$$X = c_1X_1(t) + c_2X_2(t),$$

where c_1, c_2 are arbitrary constants.

Example 3.6. The two vector solutions

$$X_1(t) = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}, \qquad X_2(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

were determined to be linearly independent solutions to

$$X'(t) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X(t).$$

Consequently, the general solution to this matrix DE is given by

$$X(t) = c_1 \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}.$$

△

3.6 Solving Systems of DEs

We finally learn how to solve a matrix DE (and hence a system of DEs)

$$X'(t) = AX(t).$$

This matrix DE should remind you of one of the very first ODEs you learn: $y' = ay$. The solution to this ODE was given by $y = ke^{at}$, where k is a constant. With this in mind, we might guess a solution to $X'(t) = AX(t)$ might take the form $X(t) = Ke^{\lambda t}$, where K is a constant vector.

By observing

$$\begin{aligned} X'(t) &= (Ke^{\lambda t})' = \lambda K e^{\lambda t} \\ AX(t) &= AKe^{\lambda t}, \end{aligned}$$

one gets the following eigenvalue problem

$$AKe^{\lambda t} = \lambda K e^{\lambda t} \iff AK = \lambda K.$$

That is to say:

> *If $X(t) = Ke^{\lambda t}$ solves $X'(t) = AX(t)$, then λ is an eigenvalue of A, and K is an eigenvector of A.*

This observation will be used to construct the general solution. However, the general solution will depend on three cases:

1. A has two distinct eigenvalues, and hence we can find two linearly independent eigenvectors;
2. A has a repeated eigenvalue, and we can find two linearly independent eigenvectors;
3. A has a repeated eigenvalue, and we can find only one linearly independent eigenvector.

3.6.1 Distinct Eigenvalues Case

Suppose A has the eigendata:

Eigenvalues:	λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$
Eigenvectors:	K_1 and K_2 corresponding to λ_1 and λ_2, respectively

Then the general solution to $X'(t) = AX(t)$ is

$$X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t}.$$

Remark: $\lambda_1, \lambda_2, K_1, K_2$ are all allowed to be complex; i.e., λ_1 and λ_2 can be complex distinct eigenvalues.

Example 3.7. *Solve*

$$X'(t) = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix} X(t)$$

First find A's eigendata:
Eigenvalues:

$$\begin{aligned}
\det A &= -2 \\
\lambda_1 &= \frac{3+\sqrt{9-4(-2)}}{2} = \frac{3+\sqrt{17}}{2} \\
\lambda_2 &= \frac{3-\sqrt{9-4(-2)}}{2} = \frac{3-\sqrt{17}}{2}.
\end{aligned}$$

Eigenvectors:
For $\lambda_1 = \frac{3+\sqrt{17}}{2}$:

$$\begin{aligned}
\begin{array}{l} (a-\lambda_1)x_1 + bx_2 = 0 \\ cx_1 + (d-\lambda_1)x_2 = 0 \end{array} &\iff \begin{array}{r} \frac{1-\sqrt{17}}{2}x_1 - 2x_2 = 0 \\ -2x_1 + \frac{-1-\sqrt{17}}{2}x_2 = 0 \end{array} \\
&\iff x_1 = \frac{1-\sqrt{17}}{4}x_2.
\end{aligned}$$

Therefore, if we take $x_2 = 4$, then $x_1 = -1-\sqrt{17}$, and so

$$K_1 = \begin{pmatrix} -1-\sqrt{17} \\ 4 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_1 = \frac{3+\sqrt{17}}{2}$.
For $\lambda_2 = \frac{3-\sqrt{17}}{2}$:

$$\begin{aligned}
\begin{array}{l} (a-\lambda_2)x_1 + bx_2 = 0 \\ cx_1 + (d-\lambda_2)x_2 = 0 \end{array} &\iff \begin{array}{r} \frac{1+\sqrt{17}}{2}x_1 - 2x_2 = 0 \\ -2x_1 + \frac{-1+\sqrt{17}}{2}x_2 = 0 \end{array} \\
&\iff x_1 = \frac{-1+\sqrt{17}}{4}x_2.
\end{aligned}$$

Therefore, if we take $x_2 = 4$, then $x_1 = -1+\sqrt{17}$, and so

$$K_2 = \begin{pmatrix} -1+\sqrt{17} \\ 4 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_2 = \frac{3-\sqrt{17}}{2}$.
General solution:

$$\begin{aligned}
X(t) &= c_1K_1e^{\lambda_1 t} + c_2K_2e^{\lambda_2 t} \\
&= c_1\begin{pmatrix} -1-\sqrt{17} \\ 4 \end{pmatrix} e^{\frac{3+\sqrt{17}}{2}t} + c_2\begin{pmatrix} -1+\sqrt{17} \\ 4 \end{pmatrix} e^{\frac{3-\sqrt{17}}{2}t}.
\end{aligned}$$

△

Example 3.8. *Solve*

$$X'(t) = \begin{pmatrix} 2 & 2 \\ -4 & -2 \end{pmatrix} X(t).$$

First find A's eigendata:
Eigenvalues:

$$\begin{aligned} \det A &= 4 \\ \lambda_1 &= \frac{0 + \sqrt{0^2 - 4(4)}}{2} = \frac{\sqrt{-16}}{2} = 2i \\ \lambda_2 &= -2i \\ K_1 &= \begin{pmatrix} \lambda_1 - (-2) \\ -4 \end{pmatrix} = \begin{pmatrix} 2i - 2 \\ -4 \end{pmatrix} \\ K_2 &= \begin{pmatrix} -2i - 2 \\ -4 \end{pmatrix}. \end{aligned}$$

Eigenvectors:
For $\lambda_1 = 2i$:

$$\begin{aligned} (a-\lambda_1)x_1 + bx_2 = 0 \\ cx_1 + (d-\lambda_1)x_2 = 0 \end{aligned} \iff \begin{aligned} (2-2i)x_1 + 2x_2 = 0 \\ -4x_1 + (-2-2i)x_2 = 0 \end{aligned}$$
$$\iff x_1 = \frac{-1-i}{2} x_2.$$

Therefore, if we take $x_2 = 2$, then $x_1 = -1 - i$, and so

$$K_1 = \begin{pmatrix} -1 - i \\ 2 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_1 = 2i$.
For $\lambda_2 = -2i$:

$$\begin{aligned} (a-\lambda_2)x_1 + bx_2 = 0 \\ cx_1 + (d-\lambda_2)x_2 = 0 \end{aligned} \iff \begin{aligned} (2+2i)x_1 + 2x_2 = 0 \\ -4x_1 + (-2+2i)x_2 = 0 \end{aligned}$$
$$\iff x_1 = \frac{-1+i}{2} x_2.$$

Therefore, if we take $x_2 = 2$, then $x_1 = -1 + i$, and so

$$K_2 = \begin{pmatrix} -1 + i \\ 2 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_1 = 2i$.
General solution:

$$\begin{aligned} X(t) &= c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} \\ &= c_1 \begin{pmatrix} -1 - i \\ 2 \end{pmatrix} e^{2it} + c_2 \begin{pmatrix} -1 + i \\ 2 \end{pmatrix} e^{-2it}. \end{aligned}$$

△

3.6.2 Repeated Eigenvalues

This situation is best framed in the context of an IVP. Hence, we let

$$K_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

always denote an arbitrary initial value for the given matrix DE. Note that the arbitrary constants x_0, y_0 will take the role of arbitrary constants c_1, c_2 from previous considerations.

Suppose A has a single repeated eigenvalue λ. Then the general solution to $X'(t) = AX(t)$ is

$$X(t) = K_0 e^{\lambda t} + K_1 t e^{\lambda t}$$

where

$$\begin{aligned} K_0 &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ K_1 &= (A - \lambda I)K_0. \end{aligned}$$

Remark. If to λ corresponds two linearly independent eigenvectors, then every nonzero vector is an eigenvector of A (why?). Therefore,

$$K_1 = (A - \lambda I)K_0 = 0,$$

and the general solution will take the form $X(t) = K_0 e^{\lambda t}$.

If you don't want to consider the IVP case, then we can obtain the general solution as follows:

$$X(t) = Kte^{\lambda t} + Pe^{\lambda t} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} te^{\lambda t} + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e^{\lambda t},$$

where P solves the system

$$\begin{aligned} (a - \lambda)p_1 + bp_2 &= k_1 \\ cp_1 + (d - \lambda)p_2 &= k_2 \end{aligned}$$

Example 3.9. *Solve*

$$X'(t) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} X(t).$$

First we find A's eigendata:

Eigenvalues:

$$\begin{aligned} \det A &= 4 \\ \lambda &= \frac{4 \pm \sqrt{16 - 4(4)}}{2} = 2 \end{aligned}$$

Therefore, A has a single repeated eigenvalue. Hence, we use the formula

$$X(t) = K_0 e^{\lambda t} + K_1 t e^{\lambda t}$$

to find the general solution.

General solution:

Let

$$K_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

denote the arbitrary initial value. Then

$$\begin{aligned} K_1 &= (A - \lambda I)K_0 \\ &= \left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) K_0 \\ &= 0. \end{aligned}$$

Therefore, the general solution is given by

$$\begin{aligned} X(t) &= K_0 e^{\lambda t} \\ &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{2t}. \end{aligned}$$

△

Example 3.10. *Solve*

$$X'(t) = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} X(t)$$

First we find A's eigendata:

Eigenvalues:

$$\begin{aligned} \det A &= 9 \\ \lambda &= \frac{6 \pm \sqrt{36 - 4(9)}}{2} = 3. \end{aligned}$$

Therefore, A has a single repeated eigenvalue. Hence, we use the formula

$$X(t) = K_0 e^{\lambda t} + K_1 t e^{\lambda t}$$

to find the general solution.

General solution:

Let

$$K_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

denote the arbitrary initial value. Then

$$\begin{aligned} K_1 &= (A - \lambda I)K_0 \\ &= \left(\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right) K_0 \\ &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} 2y_0 \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, the general solution is given by

$$\begin{aligned} X(t) &= K_0 e^{\lambda t} + K_1 t e^{\lambda t} \\ &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{3t} + \begin{pmatrix} 2y_0 \\ 0 \end{pmatrix} t e^{3t} \end{aligned}$$

△

3.7 Problems

Problem 3.1. *Convert the following systems of ODEs into matrix DEs:*

(a)

$$2x_1(t) - 2x_2(t) = x_1'(t)$$
$$-x_1(t) + -7x_2(t) = x_2'(t)$$

(b)

$$x_1'(t) = 10x_1(t) - 10x_2$$
$$x_2'(t) = 3x_2(t)$$

(c)

$$x_1(t) - x_2(t) - x_1'(t) = 0$$
$$2x_2(t) + x_1(t) - x_2'(t) = 0$$

Problem 3.2. *Solve the following matrix DEs*

(a) $X'(t) = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} X(t)$

(b) $X'(t) = \begin{pmatrix} -2 & 1 \\ 7 & 8 \end{pmatrix} X(t)$

(c) $X'(t) = \begin{pmatrix} 2 & 8 \\ -10 & -2 \end{pmatrix} X(t)$

(d) $X'(t) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} X(t)$

Problem 3.3. *Let a and b be arbitrary real numbers. Solve the following matrix DEs:*

(a) $X'(t) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} X(t)$

(b) $X'(t) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} X(t)$

(c) $X'(t) = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} X(t)$

Problem 3.4. *Let $X_1(t), X_2(t)$ be solutions to $X'(t) = AX(t)$. Show by direct computation that*

$$2X_1(t) - 3X_2(t)$$

is also a solution.

Problem 3.5. *In each part below, suppose $X_1(t), X_2(t)$ are solution vectors to some respective matrix DE $X'(t) = AX(t)$. Determine if each pair may form a general solution to their respective matrix DE.*

(a)

$$X_1(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t}, \qquad X_2(t) = \begin{pmatrix} 7 \\ \frac{21}{2} \end{pmatrix} e^{2t}$$

(b)

$$X_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}, \qquad X_2(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$$

(c)

$$X_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}, \qquad X_2(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$$

Problem 3.6. *Let*

$$K_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

be an eigenvector to a matrix A, and let $\lambda_1 = 2$ be the corresponding eigenvalue; i.e, $AK_1 = 2K_1$.

Show by direct computation that

$$X_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t}$$

cannot be a solution to $X'(t) = AX(t)$.

Problem 3.7. *Set*

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}, \qquad X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t}.$$

Convince yourself that $X_1(t)$ and $X_2(t)$ are linearly independent vector-valued functions. Despite this, use Problem 3.7 to conclude that

$$X(t) = c_1 X_1(t) + c_2 X_2(t)$$

can never be the general solution to any matrix DE $X'(t) = AX(t)$.

Problem 3.8 (Optional)**.** *Construct matrix DE's corresponding to the solution vectors given in Problem 3.5.*

Solutions to Problems

Chapter 2

You can run these through, e.g., WolframAlpha.

Chapter 3

Note: For many of these, I just ran it through Wolfram Alpha. If your answers are a little different, recall that eigenvectors are not unique. Thus, to see if your answer is correct, check if your proposed eigenvectors are just multiples of mine. On the other hand, eigenvalues are unique.

Problem 3.1

(a)

$$X'(t) = \begin{pmatrix} 2 & -2 \\ -1 & -7 \end{pmatrix} X$$

(b)

$$X'(t) = \begin{pmatrix} 10 & -10 \\ 0 & 3 \end{pmatrix} X(t)$$

(c)

$$X'(t) = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} X(t)$$

Problem 3.2

(a)

$$X(t) = c_1 \begin{pmatrix} 1+\sqrt{3} \\ 2 \end{pmatrix} e^{(2+\sqrt{3})t} + c_2 \begin{pmatrix} 1-\sqrt{3} \\ 2 \end{pmatrix} e^{(2-\sqrt{3})t}$$

(b)

$$X(t) = c_1 \begin{pmatrix} -5+4\sqrt{2} \\ 7 \end{pmatrix} e^{(3+4\sqrt{2})t} + c_2 \begin{pmatrix} -5-4\sqrt{2} \\ 7 \end{pmatrix} e^{(3-4\sqrt{2})t}$$

(c)

$$X(t) = c_1 \begin{pmatrix} -1-i\sqrt{19} \\ 5 \end{pmatrix} e^{2i\sqrt{19}t} + c_2 \begin{pmatrix} -1+i\sqrt{19} \\ 5 \end{pmatrix} e^{-2i\sqrt{19}t}$$

(d) We find $\lambda = 2$, and so A has a repeated eigenvalue. Compute

$$K_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}.$$

Therefore, the general solution is

$$X(t) = K_0 e^{2t} + K_1 t e^{2t}$$

Problem 3.3

(a)

$$X(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{at} +$$

(b) $\lambda = a$ and

$$K_1 = \begin{pmatrix} by_0 \\ 0 \end{pmatrix}$$

and so

$$X(t) = K_0 e^{at} + K_1 t e^{at}.$$

(c) $\lambda = a$ and

$$K_1 = \begin{pmatrix} 0 \\ ax_0 \end{pmatrix}$$

and so

$$X(t) = K_0 e^{at} + K_1 t e^{at}.$$

Problem 3.4

$$(2X_1(t) - 3X_2(t))' = 2X_1'(t) - 3X_2'(t) = 2AX_1(t) - 3AX_2(t) = A(2X_1(t) - 3X_2(t)).$$

Problem 3.5

(a) X_1 and X_2 are linearly dependent. So they cannot form general solution.

(b) Yes. In particular, one can take

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

(c) Yes. In particular, one can take

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Problem 3.6

By definition, $AK_1 = 2K_1$. Now, if $X_1(t)$ is a solution, we must have

$$X_1'(t) = AX_1(t) \iff 3K_1 e^{3t} = AK_1 e^{3t} \iff 3K_1 = AK_1.$$

Therefore, if $X_1(t)$ were a solution, both $AK_1 = 2K_1$ and $AK_1 = 3K_1$ would have to hold. But this is impossible since $K_1 \neq 0$.

Problem 3.7

The vector-valued functions $X_1(t)$ and $X_2(t)$ are linearly independent because it's impossible to find a constant c such that $e^{2t} = ce^{3t}$. Now, for sake of contradiction, suppose that $X(t) = c_1X_1(t) + c_2X_2(t)$ is the general solution to a matrix DE $X'(t) = AX(t)$. Then, $X_1(t)$ and $X_2(t)$ are solutions to $X'(t) = AX(t)$. But

$$X_1(t) = Ke^{2t} \text{ a solution } \implies (Ke^{2t})' = AKe^{2t} \implies 2K = AK$$
$$X_2(t) = Ke^{3t} \text{ a solution } \implies (Ke^{3t})' = AKe^{3t} \implies 3K = AK.$$

But both $2K = AK$ and $3K = AK$ cannot possibly hold, and therefore, $X_1(t)$ and $X_2(t)$ cannot both be solutions. Therefore, $X(t) = c_1X_1(t)+c_2X_2(t)$ cannot be the general solution lest we arrive at a contradiction.

Problem 3.8

See solution to Problem 3.5.