**3.2 Rates of Change**

- Average Rate of Change
- Instantaneous Rate of Change
- Tangent Line

**Exam 1:**

- Monday 2/25/19

**WebAssign on this section:**

- Sunday 2/24/19

**Warm Up:**

- Evaluate the following limit using algebraic techniques:

$$\lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4}$$

## Average Rate of Change

Definitions:

- Average Rate of Change on the Interval $[a, b]$

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(b) - f(a)}{b - a}$$

- Average Rate of Change

The average rate of change of $y = f(x)$ with respect to $x$ from $a$ to $b$ is the quotient

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

- Units of Average Rate of Change

$$\text{units of average rate of change} = \frac{\text{units of output}}{\text{units of input}} = \frac{\text{units of } f(x)}{\text{units of } x}$$

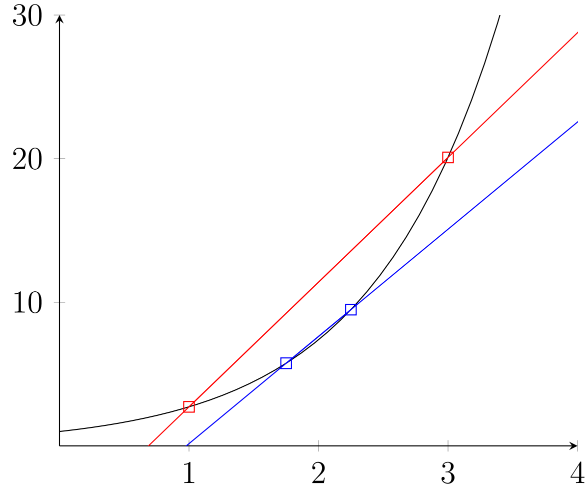
- Secant Line

The line that passes through two points $(a, f(a))$ and $(b, f(b))$

- Average Rate of Change Is Slope of Secant Line

The average rate of change of $y = f(x)$ from $a$ to $b$ is the slope of the secant line from $(a, f(a))$ to the point $(b, f(b))$.

**Example of Secant Lines**



**Examples**

Let $f(x) = x^2$. Find the average rate of change over the intervals $[1, 2]$, $[1, 1.5]$, and $[1, 1.1]$. Sketch a graph of the function and draw in secant lines for each of those intervals. Label each secant line with its slope.

**Examples**

Calculate the average rate of change of the given function over the given interval. Specify the units of measurement. Round your answer to three decimal places.

Interval: $[2, 8]$

| $t$ (months) | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|
| $R(t)$ (\$) | 20,600 | 24,300 | 20,000 | 19,900 | 22,800 | 19,300 |

**Do On Your Own**

The cost (in dollars) of producing $x$ units of a certain commodity is given below.

$$C(x) = 7000 + 12x + 0.15x^2$$

Find the average rate of change of C with respect to $x$ when the production is changed from $x = 100$ to the given value.

$$x = 104$$

# Instantaneous Rate of Change

Definitions

- Average Velocity

> Suppose $s = s(t)$ describes the position of an object at time $t$. The average velocity from $a$ to $a + h$ is
> $$\text{average velocity} = \frac{s(a+h) - s(a)}{h}$$

- Instantaneous Velocity

> The instantaneous velocity (or simple velocity) $v(a)$ at time $a$ is
> $$v(a) = \lim_{h \to 0}(\text{average velocity}) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$
> if this limit exists

- Instantaneous Rate of Change

> Given a function $y = f(x)$, the instantaneous rate of change of $y$ with respect to $x$ at $x = a$ is given by
> $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
> if this limit exists

**Types of Problems**

Be sure that you can find the instantaneous rate of change (using the limit definition) for the following types of functions:

- Linear (i.e. $y = ax + b$)
  - $y = x$
  - $f(x) = 3x + 2$
  - $C(x) = 7000 + 12x$
- Quadratic (i.e. $y = ax^2 + bx + c$)
  - $y = x^2$
  - $f(x) = x^2 + 2x + 1$
  - $p(x) = (x + 1)^2 + 3$
  - $C(x) = 7000 + 12x + 0.15x^2$
- Cubic (i.e. $y = ax^3 + bx^2 + cx + d$)
  - $y = x^3$
  - $f(x) = x^3 + 3x^2 + 3x + 1$
  - $p(x) = 2(x + 1)^3 + 2$
  - $C(x) = 7000 + 12x + 0.15x^2 + 0.009x^3$
- $\frac{1}{\text{Linear}}$ $\left(\text{i.e. } y = \frac{1}{ax + b}\right)$
  - $y = \frac{1}{x}$
  - $f(x) = \frac{1}{3x + 2}$
  - $C(x) = \frac{1}{7000 + 12x}$
- $\sqrt{\text{Linear}}$ (i.e. $y = \sqrt{ax + b}$)
  - $y = \sqrt{x}$
  - $f(x) = \sqrt{3x + 2}$
  - $C(x) = \sqrt{7000 + 12x}$

**Examples**

Let $f(x) = \ln(x)$. Find the average rate of change on the intervals $[2, 2+h]$ and $[2-h, 2]$ for several small values of $h$. Use this to estimate the instantaneous rate of change at $x = 2$.

**Examples**

Find the instantaneous rate of change of $f(x) = x^2 + 3x$ at $x = 3$, using algebraic techniques for limits. (You must use the limit definition for instantaneous rate of change).

**Do On Your Own**

Find the instantaneous rate of change of $f(x) = x^2$ at $x = 1$, using algebraic techniques for limits. (You must use the limit definition for instantaneous rate of change).

## Tangent Line

Definitions

- Tangent Line

> The tangent line to the graph of $y = f(x)$ at $x = c$ is the line through the point $(c, f(c))$ with slope
> $$m_{\tan}(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$
> provided that this limit exists.

- Instantaneous Rate of Change and the Slope of the Tangent Line

> If the instantaneous rate of change of $f(x)$ with respect to $x$ exists at a point $c$, then it is the slope of the tangent line at that point.

**Examples**

Find the equation for the tangent line to $f(x) = x^2$ at $x = 1$. Graph both $f(x)$ and its tangent line on the same graph.

**Do On Your Own**

Find the equation for the tangent line to $f(x) = x^3 - 2x$ at $x = 1$. Graph both $f(x)$ and its tangent line on the same graph.